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Lyapunov-based Model Predictive Control of Nonlinear Quadruple Tank System Using Constrained and Unconstrained Methods

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ABSTRACT

Model predictive control is a widely utilized advanced control technique in industrial settings, valued for its capacity to effectively manage complex multi-input multi-output processes while optimizing process performance. Lyapunov-based model predictive control of Nonlinear Systems is a method that incorporates the objective function as a suitable Lyapunov function, thereby ensuring stability of the nonlinear system. This study involves the modeling of a nonlinear system to the Lyapunov function and the analysis of a quadruple tank using nonlinear model predictive control and barrier function-based model predictive control algorithms. Additionally, a comparison is made between a PID controller and a well-established industrial controller to evaluate the performance of the nonlinear quadruple tank, with the findings indicating satisfactory results.

1. Introduction

The concept of Model Predictive Control (MPC) was first introduced in the late 1970s and has since undergone significant development. MPC is a control strategy that leverages a dynamic model of the system to forecast its future behavior and optimize control actions within a finite time frame. Nonlinear MPC is an advanced control methodology employed in diverse engineering contexts characterized by nonlinear system dynamics. Nonlinear MPC builds upon the MPC framework to accommodate systems with nonlinear dynamics, thereby expanding its applicability to a wider spectrum of engineering applications. Barrier function-based MPC is a specialized form of MPC that integrates barrier functions to address constraints within the optimization problem. Barrier functions are mathematical functions created to penalize breaches of constraints, enabling the optimization algorithm to maneuver around constraint boundaries. This approach is especially beneficial for managing systems that are subject to both equality and inequality constraints.

The stability of the problem is considered an essential aspect of MPC. For instance, if the feasibility condition is not included in the MPC theorem, it may yield incorrect results and lead to issues such as model faults,

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disturbances, and incorrect initial conditions. Furthermore, if the associated system is not stable, the controller's actions will not converge to the desired outcome. Therefore, addressing stability and feasibility is crucial in resolving MPC problems. One proposed solution to achieve stability and feasibility is to incorporate a cost function, such as a Lyapunov function, which ensures the fulfillment of the two aforementioned criteria for a closed-loop system.

In a previous studies, [1] suggested the combination of Koopman operator methodology with Lyapunov-based MPC to stabilize nonlinear systems. The Koopman operator allows for linear representations of nonlinear dynamic systems by transforming their dynamics into a higher-dimensional space using observable functions. These functions are then controlled by the linear Koopman operator in an infinite-dimensional space. Ghasemi et al in [2] introduced a model predictive controller for a multilevel asymmetric cascade grid-connected inverter, aiming to predict and control system behavior several steps ahead. The controller was developed using the Lyapunov theory as a basis for the design of a predictive function controller. A barrier function-based MPC approach that balances safety and performance is discussed in [3]. This method combines the control barrier function and the control Lyapunov function in a time-invariant MPC framework without constraints, guaranteeing both safety and optimal decision-making. In [4], a MPC utilizing an ensemble of recurrent neural network models is presented, incorporating control Lyapunov-Barrier functions to ensure closed-loop stability and operational safety for nonlinear processes in the presence of two categories of unsafe regions: bounded and unbounded sets. Bui et al. in [5] introduce a novel Lyapunov-based nonlinear MPC approach for addressing the attitude control challenge in unmanned aerial vehicles, a critical aspect of their operational functionality. The proposed controller is formulated using a quadratic cost function that incorporates the dynamics of the unmanned aerial vehicles, and system constraints. Furthermore, an additional contraction constraint is incorporated to guarantee the stability of the closed-loop system. The author in [6] presents three detection strategies derived from a Lyapunov-based economic MPC for nonlinear systems. The initial approach involves introducing randomized adjustments to an Lyapunov-based economic MPC formulation in real-time to potentially identify cyberattacks. The second approach includes identifying attacks by comparing the variance between state measurements and state forecasts with a set threshold. Finally, the third tactic uses duplicate state estimators to recognize deviations from the anticipated standard process behavior as potential cyber threats. A study conducted by Aminsafaee and colleagues in [7] addresses the issue of robust stabilization in a specific category of nonlinear discrete-time switched systems characterized by polytopic uncertainties and unknown state delay. Additionally, the study considers the presence of constraints on the control signal. The main goal of the suggested controller is to stabilize the switched system when faced with any type of switching signals, by employing the approach of switched Lyapunov function. In [8], a method of predictive direct power control is formulated using the Lyapunov function approach for regulating the grid-connected photovoltaic converter. The predictive control methodology employs a discretized representation of the photovoltaic converter to anticipate forthcoming active and reactive power levels within the system. This process involves evaluating a cost function across various voltage vectors, with the algorithm subsequently identifying the most advantageous voltage vector that minimizes the cost function. This selection enables the calculation of model variables for the subsequent sampling period. In [9], a continuous-time Barrier Function-based MPC approach is introduced for linear systems subject to Signal Temporal Logic specifications and input constraints. The fulfillment of the Signal Temporal Logic specifications is represented using time-varying barrier functions, and a control law is formulated to minimize violations in cases where the specified task with a given robustness level cannot be achieved, such as due to limitations in actuation. The proposed scheme ensures recursive feasibility by incorporating a time-varying terminal constraint, which enforces a desired system behavior to ensure the fulfillment of the task with predetermined robustness. A Lyapunov-based MPC approach was suggested in [10] to enhance shipboard boom cranes by incorporating ship rolling, optimizing, and addressing input saturation. This includes incorporating ship rolling into the shipboard boom crane model's initial state variables to aid in the development of subsequent controllers. The author of [11] concentrates on the task of guiding autonomous underwater vehicles through challenging ocean environments. They have developed a new MPC framework based on Lyapunov principles, which enhances the vehicles' ability to track trajectories by continuously optimizing their performance in real time. In [12], the effectiveness of MPC and PID methods was compared. This article also examines the comparison between the two methods to assess the effectiveness of MPC. Our study is based on a similar model and evaluates the effectiveness of MPC by comparing it with PID after implementing the desired MPC methods. The authors in [13] delve deeper into the potential of Lyapunov-based Economic MPC to achieve various goals during process operation. This includes assisting in distinguishing between mechanistic models in real time. Specifically, when multiple competing mechanistic models can explain the available data, a new series of "online experiments" can be carried out to gather additional information and eliminate models that do not accurately represent the actual process. In reference [14], for arbitrarily constant or piecewise-constant set-points and disturbances, it has been proposed to use a nonlinear model predictive controller to accomplish offset-free tracking and disturbance rejection. The controller deals with the control issue by managing the non-linear behavior of the plant and integrating the tracking error of the controlled variables. This approach offers a simple way to track and counteract disturbances resulting from unknown set-points or disturbances. The study in [15] examines the control of a system with Slow-Rate Instantaneous Measurement for the first time. It introduces a pole placement state feedback with feedforward controller to enable both Slow-Rate Instantaneous Measurement and traditional fast-rate output to follow their step reference input. This controller includes a periodically changing gain and an observer to estimate the slow-rate instantaneous state variable at sampling points. In [16], a simulation of a quadruple tank system is used to illustrate common outcomes. The suggested controller design features a simplified ordering structure that employs a set of designated linear transfer functions and gains to minimize a Generalized Predictive Control cost index. This methodology allows for the integration of various traditional controller setups within the feedback loop, including extended PI, PID, Lead-lag, or a more comprehensive transfer function framework. Pazoki et al [17] present a comparative analysis of the performance of various algorithms based on their performance characteristics and indices. The evaluation of these algorithms is conducted through multiple simulation series, demonstrating that State Space MPC exhibits superior performance, as well as robustness against disturbances and noise. Therefore, State Space MPC is deemed the most suitable choice for implementation in the context of a Continuous Stirred Tank Reactor. The scholars in reference [18] aim to create and assess an Adaptive Pole Placement Controller and a resilient Adaptive Sliding Mode Controller to efficiently regulate a minimum phase Quadruple Tank Process. The controllers are evaluated using simulations, and their effectiveness is measured against a PID controller in terms of how well they handle changes in set points, reject disturbances, and cope with uncertainties in parameters. In [19], two novel adaptations of modified active disturbance rejection control are introduced to stabilize a nonlinear quadruple tank system and regulate the water levels of the lower two tanks in the face of external disturbances, uncertain parameters, and varying input set-points. A study in [20] presents various alternatives for MPC by comparing different implementations of MPC. This comparative analysis will be conducted internally and demonstrated using the four-tank benchmark, a well-researched system often examined by scholars focusing on industrial applications. Sorcia-Vázquez and colleagues [21] introduce a decentralized MPC design for nonlinear systems, taking into account the interaction between control inputs. This controller utilizes a centralized robust tube-based nonlinear MPC system. A significant development lies in the formulation of a method to partition the process model into s subsystems, enabling the design of robust tube-based controllers that guarantee a constrained linearization error. In [22], a study presented the development and implementation of an advanced MPC controller and a PID controller within a programmable logic controller for a quadruple tank process. The study involved a comparative analysis of two controllers that were designed according to the specified initial process conditions, which mandated valve opening values to fall between 60% and 80% to enable effective communication between interconnected tanks. The PID controllers were programmed in Ladder language with PID blocks within Tia Portal V16 for process regulation, whereas the MPC controller was formulated using structured language SCL exported from Matlab-Simulink to Tia Portal V16. In [23], the authors tackled the previously mentioned difficulties by developing control strategies to manage the water flow in the lower two tanks of the quadruple tank system. To compare and assess performance, three distinct controller algorithms - a nonlinear Model Predictive Control (MPC), a nonlinear MPC with an extended Kalman filter, and a linear MPC - were investigated in the study and development of the control system for a quadruple water level system under non-minimum phase conditions using the MATLAB simulation platform. Within model predictive control, the paper [24] an engineering perspective' offers a meticulous examination encompassing theoretical underpinnings, historical trajectories, and pragmatic insights. Notably, the discourse delves into one of the pivotal challenges confronting MPC implementation: the computational burden. Through a judicious analysis, the paper elucidates how managing computational complexity stands as a cornerstone in the design and deployment of MPC systems, thereby providing valuable guidance for engineers and researchers navigating this intricate terrain.

The book [25] explores new developments in MPC, especially in economic and distributed MPC. It not only discusses the latest advancements but also raises important questions for further research and practical use. By addressing both theoretical and practical challenges, this book guides researchers and practitioners towards better understanding and application of MPC techniques, fueling innovation in the field.

Consequently, we constructed a Lyapunov function using nonlinear MPC and barrier function MPC algorithms in a quadruple tank system, and conducted an analysis. The aim of this study is to compare the effectiveness of constrained and unconstrained MPC methods in nonlinear systems. Furthermore, we verified the simulation by incorporating a PID controller to demonstrate the efficacy of our proposed approach in nonlinear systems. We have rephrased the section detailing our contributions to explicitly emphasize the novelty and significance of our work. Specifically, we have:

- Develop a Lyapunov function using nonlinear MPC algorithm for quadruple-tank
- Applied a Lyapunov function using barrier function MPC algorithm for quadruple-tank
- Comparison the suggested approaches with a PID controller and analyzing them

The remainder of this paper is organized as follows: section 2 is described the problem definition. It uses nonlinear MPC and barrier function-based MPC. Section 3 introduces quadruple tanks and related parameters as a case study. Section 4 shows the simulation results and analyzes them. Finally, Section 5 concludes the paper.

2. Problem Definition

Nonlinear systems face significant challenges due to uncertainty and disturbances, making traditional linear control methods ineffective. Ensuring stability and feasibility in these systems is crucial for control problems, as instability can lead to undesirable controller responses and model errors. To address this, incorporating Lyapunov functions into the objective function problems has been proposed as a solution to guarantee stability and feasibility, leading to the development of a branch of model predictive control known as Lyapunov function-based MPC. This approach has spawned various algorithms tailored to specific problem features. In practical applications, the importance of nonlinear control and MPC is clear. Designing an MPC controller for nonlinear systems is a highly intricate task, especially if the system is unstable. The suggested controllers should be able to characterize the system's stability and demonstrate the specific scenario within an unchanging set. Therefore, the MPC, based on the Lyapunov problem, becomes an optimization challenge. The subsequent section outlines the formulation of the MPC based on the Lyapunov approach. A discrete nonlinear system is regarded as follows:

$$x_{k+1} = F(x_k, u_k) \tag{1}$$

where u is the vector of control inputs, k is sampling time, F is the evolution operator representing the dynamics that transfer the system states forward in time, and x is the vector of state variables sampled discretely in time. The Lyapunov theorem defines MPC-based Lyapunov as follows:

$$\min_{u_k,\dots,u_{k+N_u+1}} \sum_{j=1}^{N_p} \|x_{k+j}\|_W^2 + \sum_{j=1}^{N_u} \|x_{k+j-1}\|_R^2 \tag{2}$$

$$S.t.\ \ \hat{x}_{k+j} = F\left(x_{k+j-1}, u_{k+j-1}\right),\quad j = 1, \dots, N_p \tag{3}$$

$$\hat{x}_k = x_k \tag{4}$$

$$V\left(\hat{x}_{k+j}\right) \le r, \quad j = 1, \dots, N_p \tag{5}$$

$$V(\hat{x}_{k+1}) - V(\hat{x}_k) \le V\left(F\left(x_k, h(x_k)\right)\right) - V(x_k) \tag{6}$$

Where $\hat{x}_{k+j}$ is the predicted state trajectory with initial (measured) state x_k, u_{k+j} denote the calculated manipulated input variables j time steps ahead and N_p, N_u denotes the prediction and control horizons, respectively. The operator $\|.\|_Q^2$ denotes the weighted Eucledian norm defined for an arbitrary vector x and weighting matrix Q as $\| x \|_Q^2 = x^T Q x$ and $W \in \Re^{n\times n}, R \in \Re^{m\times m}$ denote the positive definite weighting matrices for the state and input vectors respectively. Furthermore, V(x) is the Lyapunov function associated with the explicit control law h(x). In the following, we will define nonlinear MPC and barrier function-based MPC algorithms to apply them in our problem.

2.1. Nonlinear MPC

Suppose in a control process, $x(n)$ variable which $n = 1,2,3, \dots$ has been measured during the discrete time interval. Control process means that the way in which $u(n)$ is determined as a controllable signal could influence the procedure of the system. In tracing problems, the main goal is determining the $u(n)$ which triggers $x(n)$ follows the reference proportion of the input as much as possible. The vectors related to $x(n)$ and $u(n)$ are described as:

$$x(n) \in R^d, u(n) \in R^m \tag{7}$$

Here u(n) is considered in the form of $u(n) = \mu(x(n))$ which could resist changing the x(n) from the reference proportion $x^* = 0$. The whole equation of the systems are regarded as:

$$x^+ = f(x, u) \tag{8}$$

In this equation, $f: X \times U \to X$ is a non-linear function that transmits state variable x(n) and control signal u(n) to $x+$ (subsequent state of the system). With starting from the current state of $x(n)$ for any $u(0) \dots, u(N-1)$ and through (1) the prediction of Xu is considered as:

$$x_u(0) = x(n) \tag{9}$$

$$x_u(k+1) = f\left(x_u(k), u(k)\right), k = 0, \dots, N-1 \tag{10}$$

Now $u(0), \dots, u(N-1)$ is determined by means of optimal control theory which Xu would be nearby X^* as much as possible. For this purpose, distance between reference input and xu(k) which $k = 0, \dots, N-1$ is measured with the cost function which is $L(x_u(k)), u(k))$. The amount of difference between control signal u(k) and reference control signal also is considered in cost function which here $U^* = 0$. A kind of common Cost Function for mentioned purpose is square Cost Function that can be expressed as:

$$L(x_u(k), u(k)) = \|x_u(k)\|^2 + \lambda \| u(k) \|^2 \tag{11}$$

In (11), $\lambda \ge 0$ is a coefficient that if there are not any limitations on control signal, it is considered zero. Optimal control problem is regarded as:

$$\min J(x(n), u(0)) = \sum_0^{N-1} L(x_u(k), u(k)) \tag{12}$$

Suppose that this optimal control problem has a result that is obtained with minimizing of $u^*(0), \dots, u^*(N-1)$. In order to achieve the optimum feedback the following expression is considered.

$$\mu(x(n)) = u^*(0) \tag{13}$$

This equation (13) expresses that in any step, the first coefficient of optimal control signal sequence is exerted on the system which is the same receding horizon strategy. In fact, the rule of feedback μ is obtained by means of online optimal repeated algorithm and created predictions. In this way, constrained optimization method is used in order to exert limitations which some of its drawbacks are being more complex, increasing the probability of losing feasibility and overloading the amount of calculation. As per the provided elucidation, the cost function is delineated subsequently:

$$J_0^*(x_0) = \min_{U_0} p(x_N) + \sum_{i=0}^{N-1} q(x_i, u_i) \tag{14}$$

Where $p(x_N)$is terminal cost and $q(x_i, u_i)$is stage cost as:

$$p(x_N) = x_N^T P x_N \tag{15}$$

$$q(x_i, u_i) = x_i^T Q x_i + u_i^T R u_i \tag{16}$$

where P, Q and R are positive definite matrices.

2.2. Barrier function-based MPC

In the MPC approach utilizing barrier function, the constrained optimization problem is transformed into an unconstrained problem by incorporating additional coefficients into the cost function. This adaptation facilitates the application of unconstrained optimization techniques. Furthermore, this modification ensures compliance with all relevant conditions and addresses any associated limitations of this optimization method. In equation (1), with considering N as a result of the prediction and control, if u is considered an index for the control sequence in the form of $u = u(0), u(1), \ldots, u(N-1)$, in this case $x^u(0;x)$ shows state variables sequence (with range of N) which start of it is x state furthermore would be affected by u. So the equation for $x^u(.;x)$ is considered as:

$$x^u(0;x) = \left(x, f(x,u(0)), \ldots, f(x^u(N-1;x), u(-1))\right) \quad (17)$$

Therefore $x^u(0;x) = x$ and also N term of $x^u(0;x)$ is regarded as $x^u(i;x)$. State variables are in the convex and close set of X and inputs are in compact and convex set of U. In addition, ultimate state of the system $x_u(N;x)$ should be part of the X_f set. Also, every set which was mentioned has a nonempty subset that are showed as X^0, $X_f(0)$ and U^0. Finally, the feasible set $u_n(x)$ is described as:

$$u_n(x) = u \quad (18)$$

Where $u(i) \in U$, $x^u(i,x) \in X$, $x^u(N;x) \in X_f$, $i = 0, \ldots$,N. Barrier functions are used to ensure the forward invariance of a set and are similar to Lyapunov functions in terms of theoretical construction. A kind of function that approaches infinity and is bounded within a positive set is called a barrier function. The function guarantees forward invariance by never reaching infinity within a safe set by applying appropriate criteria to its derivative. However, a significant drawback of this approach is the complexity of calculating constrained optimization and the reduction of feasible circumstances. By modifying the Cost Function, the constrained optimization problem in the MPC approach based on the Barrier Function is changed into an unconstrained problem. This not only simplifies calculations but also addresses all design-related constraints. Cost Function for the standard MPC problem is described as:

$$min : f(x(N)) + \sum_0^{N-1} i(x(k, u(k)) \quad (19)$$

$$u(k) \in U, x(k) \in X, x(N) \in X_f \subset X \quad (20)$$

$$x(k+1) = f(x(k), u(k)), k = 0, \ldots, N-1 \quad (21)$$

Where $\mathrm{i(x,u)}$ is the stage cost which is defined as the set of square functions from input and state variables in the form of $\| x \|_Q^2 + \| x \|_R^2$ and also f(x) is the terminal cost of which a reasonable choice would be $\| x \|_P^2$. One method of incorporating constraints into MPC design is through the use of a cost function and solving the optimization problem using a constrained optimization algorithm. However, this approach presents challenges, including the complexity of calculating constrained optimization and ensuring the feasibility of the system. Therefore, in MPC techniques that rely on barrier function principles, the constrained optimization problem is converted into an unconstrained optimization problem through adjustments made to the cost function. Barrier functions are conceptually similar to Lyapunov functions and are utilized to ensure the residual property of a specified set. A suitable barrier function is characterized by its ability to maintain positive values within the set and to increase towards infinity as it approaches the boundaries of the set. Thus, by commencing with an appropriate initial condition and enforcing specific constraints on its derivative to ensure that the function within a secure set does not approach infinity, the integrity of the set is consistently maintained. The cost function of the barrier function is as follows:

Fig. 1. A schematic of quadruple tank located in Faculty of Electrical Engineering at K.N.Toosi University

$$u^*(x(k)) = argmin_{U_0} F(x_u(N;x)) + l_0(x, u(0)) + \sum_{i=0}^{N-1} l(x_u(i;x), u(i)) \quad (22)$$

$$s.t.$$

$$x(k+1) = f(x(k), u(k))$$

Where:

$$\mathrm{F(x)} = \|\mathrm{x}\|_\mathrm{P}^2 + \mu\mathrm{B_f(x)} \quad (23)$$

$$l_0(\mathrm{x,u}) = \|\mathrm{x}\|_\mathrm{Q}^2 + \|\mathrm{u}\|_\mathrm{R}^2 + \mu\mathrm{B_u(u)} \quad (24)$$

$$\mathrm{l(x,u)} = \|\mathrm{x}\|_\mathrm{Q}^2 + \|\mathrm{u}\|_\mathrm{R}^2 + \mu\mathrm{B}_x\mathrm{(x)} + \mu\mathrm{B_u(u)} \quad (25)$$

$\mathrm{B_f}$, B_x and $\mathrm{B_u}$ are barrier functions for Xf, $\mathbb{X}$ and $\mathbb{U}$. The matrices P, Q, and R are assumed to be symmetric and positive definite, and the scalar parameter μ, which is the weight of the barrier terms, is assumed to be positive.

Assumption of stability: The nonlinear systems under consideration are limited to a group of stable systems represented by $x(t)$ that adhere to the input constraints for all u(t) = h(x), ensuring the presence of a feedback control law within a consistent region and stabilizes the closed loop system asymptotically. This action is tantamount to postulating the presence of a Lyapunov function for the nominal system, as per Lyapunov's theorem [26-27].

3. Case study: Quadruple Tank

This study focuses on the quadruple tank system as a case study for experimental testing of control techniques on real systems. The system has been designed to allow for the creation of a controller at various operational points, but its nonlinear nature may hinder its performance across the entire operational range. The system's flexibility enables the implementation of different experiments, and manual valve adjustments provide access to distinct dynamics. Fig. 1 depicts a quadruple tank specimen situated within the process control laboratory of the Faculty of Electrical Engineering at K.N.Toosi University. This equipment is utilized for both educational and research endeavors.

Initially, the experimental procedure involving the quadruple tank, as employed in the present study, was outlined by Henry and Johnson. This process entails the utilization of four water tanks, with two positioned above and two below, in conjunction with two valves and two pumps. The primary objective of this setup is to regulate the water level using the pumps. The inputs for this system consist of the pump voltages, while the outputs are represented by the water levels in the two lower tanks. Building upon the work of Johnson and Henry, the nonlinear model of the system, based on Bernoulli's principle, has been expounded in previous studies [15], [16]. This principle is described as:

$$\begin{aligned}\frac{dh_1}{dt} &= -\frac{a_1}{A_1}\sqrt{2gh_1}+\frac{a_3}{A_1}\sqrt{2gh_3}+\frac{\gamma_1 k_1}{A_1}v_1\\ \frac{dh_2}{dt} &= -\frac{a_2}{A_2}\sqrt{2gh_2}+\frac{a_4}{A_2}\sqrt{2gh_4}+\frac{\gamma_2 k_2}{A_2}v_2\\ \frac{dh_3}{dt} &= -\frac{a_3}{A_3}\sqrt{2gh_3}+\frac{1-\gamma_2 k_2}{A_3}v_2\\ \frac{dh_4}{dt} &= -\frac{a_4}{A_4}\sqrt{2gh_4}+\frac{1-\gamma_1 k_1}{A_4}v_1\end{aligned} \tag{22}$$

In these equations, h_i is tank fluid height, A_i is cross section, a_i the cross section of the output. The exerted voltage on pump i equals with v_i and the fraction of current of it would be $k_i v_i$. Also $\gamma_1, \gamma_2 \in (0,1)$ are relevant to regulation of input valves. Gravitational acceleration is g and the outputs of the system which are the surface of the first and second tank are measured with $k_c h_i$. Finally, the quantities about parameters mentioned with considering the experimental condition are regarded in Table I. In order to make the system linear around the operation point, parameters are mentioned in Table II. Then with considering the mentioned quantities, the linear state equations would be defined as:

$$\frac{dx}{dt}=\begin{bmatrix}-\frac{1}{T_1} & 0 & \frac{A_3}{A_1 T_3} & 0\\ 0 & -\frac{1}{T_2} & 0 & \frac{A_4}{A_2 T_4}\\ 0 & 0 & -\frac{1}{T_3} & 0\\ 0 & 0 & 0 & -\frac{1}{T_4}\end{bmatrix}x+\begin{bmatrix}\frac{\gamma_1 k_1}{A_1} & 0\\ 0 & \frac{\gamma_2 k_2}{A_2}\\ 0 & \frac{1-\gamma_2 k_2}{A_3}\\ \frac{1-\gamma_1 k_1}{A_4} & 0\end{bmatrix}u \tag{20}$$

$$y=\begin{bmatrix}k_c & 0 & 0 & 0\\ 0 & k_c & 0 & 0\end{bmatrix}x \tag{21}$$

T_i are calculated below:

$$T_i=\frac{A_i}{a_i}\sqrt{\frac{2h_i^2}{g}}, i=1,2,3,4 \tag{22}$$

In this experiment, the height of every tank and the most water discharge through pumps are around 30 cm and 2.5 liter per minutes respectively. Therefore maximum voltage of pumps id considered around 12.5. Consequently, the maximum voltage that can be supplied to the pumps is estimated to be around 12.5 volts. These limitations are regarded as the constraints to the issue. These constraints are intended to serve as design constraints during the controller design phase. Furthermore, the desired target heights are H1 set at 27 and H2 set at 9.

$$0 \le h_i \le 30 \tag{23}$$

$$0 \le v_i \le 12.5 \tag{24}$$

4. Simulation Results

Similar to linear MPC, nonlinear MPC computes control inputs at regular intervals by utilizing a blend of model-based prediction and finite optimization. Conversely, barrier function-based MPC converts a constrained optimization problem into an unconstrained or equally constrained problem, allowing for the application of efficient optimization techniques. This study examines the simulation of two methodologies, namely nonlinear MPC and barrier function-based MPC, using MATLAB.

Table II. The quadruple tank system operating point values

Amount	Dimentions	Parameters
(12.4,12.7)	cm	(h_1^0, h_2^0)
(1.8,1.4)	cm	(h_3^0, h_4^0)
(3,3)	V	(v_1^0, v_2^0)
(3.33,3.35)	cm^3/Vs	(k_1, k_2)
(0.7,0.6)		(γ_1, γ_2)

Table III. The parameters and coefficients of the controllers and horizon times

Parameter	Value
T_P	500
T_C	1
$V_1(0)$	3
$V_2(0)$	3
μ	0.5
l_b	0
u_b	17

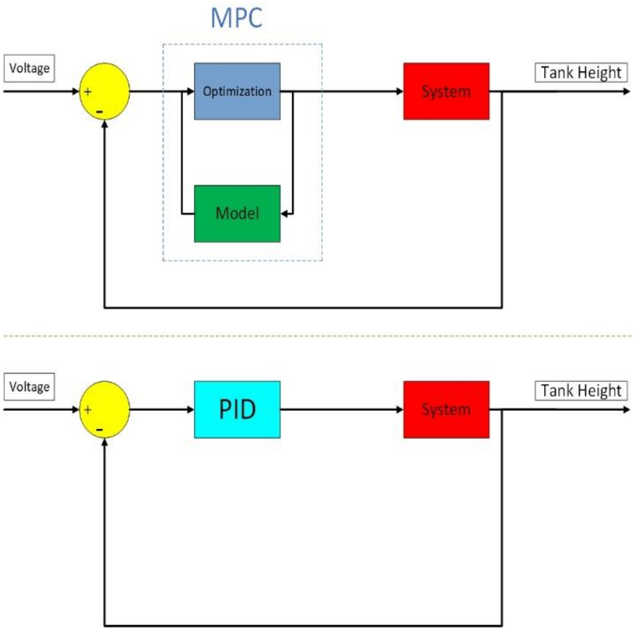

Fig.2. The schematic of the system with MPC and PID controllers

The proposed quadruple tank system considers tanks 3 and 4 as inputs to tanks 1 and 2. The nonlinear MPC algorithm is employed to address constrained nonlinear systems. Figure 2 shows the schematic of the system with MPC and PID controllers. The parameters and coefficients of the controllers and horizon times are reported in Table **III.** The outcomes of the nonlinear MPC controller and variations in the Cost Function are presented in Fig. 2 and Fig. 5. As depicted in Fig. 4, the designed controller effectively accomplishes the tracking task. Additionally, the design requirements are fully met.

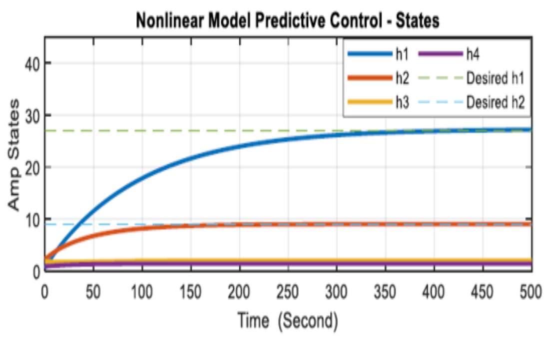

Fig. 3. Response of the nonlinear model predictive control state

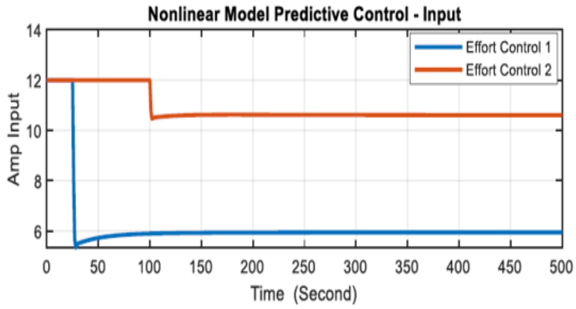

Fig. 4. Response of the nonlinear model predictive control input

The results in Figures 6 and 8 illustrate the outcomes of MPC utilizing barrier functions and the corresponding variations in the Cost Function. Upon examination of Figures 6 and 7, it is evident that all specified conditions have been met. Ultimately, the system has successfully attained the intended output. The comparison of the two approaches has revealed that the barrier function-based MPC exhibits a swifter response compared to the nonlinear method. Furthermore, the former method demonstrates a lower cost function than the nonlinear MPC approach. Conversely, the nonlinear MPC approach requires greater control effort. The findings indicate that both methods perform well in simulation and meet stability requirements. Additionally, the comparison conducted in this study serves to elucidate the distinctions between constrained and unconstrained optimizations. A PID controller is a feedback control loop utilized extensively in industrial systems. To validate our simulation, we conducted a comparison between a PID controller and our proposed MPC controller. Figure 9 illustrates the comparison between a nonlinear MPC and a PID controller. The comparison focused on Tank 1 and Tank 2, which are analogous tanks. The findings indicate that the nonlinear MPC method achieved stability more rapidly than the PID controller. Additionally, no disturbances were observed in the nonlinear MPC, whereas the PID controller exhibited disturbances leading to instability.

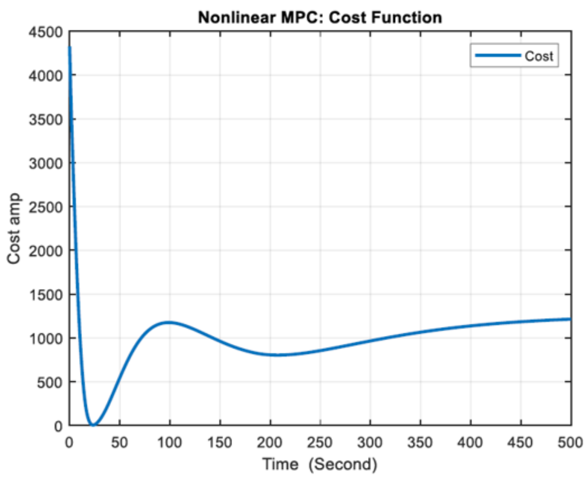

Fig. 5. Cost function of simulation in nonlinear MPC

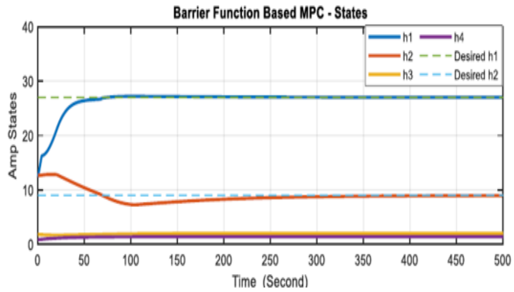

Fig. 6. Response of the barrier function based model predictive control state

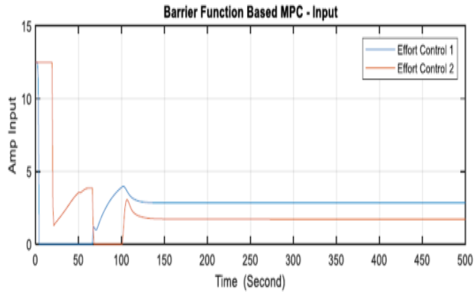

Fig. 7. Response of the barrier function based model predictive control input

5. Conclusion

This study investigates MPC methods based on Lyapunov function. Specifically, it explores two different techniques for implementing MPC in a closed-loop system with a limited-time state. These techniques include a nonlinear MPC controller that utilizes constrained optimization methods, and a barrier function-based MPC controller that introduces barrier functions to transform the constrained optimization problem into an unconstrained optimization method. The study conducts simulations to evaluate the performance of these methods in tracking the quadruple tank. The results indicate that both methods effectively track the input, but the unconstrained optimization method requires fewer calculations and incurs lower costs. Additionally, the study notes that while MPC based on Lyapunov prioritizes system stability, it may not necessarily represent the most optimized approach.

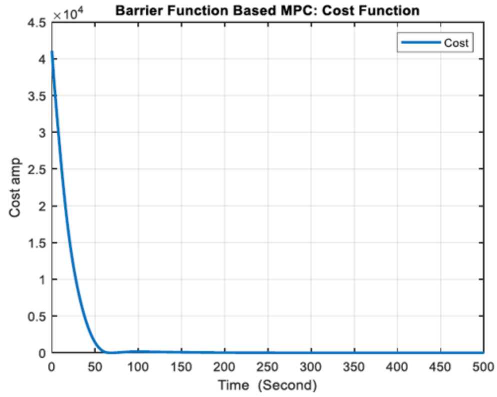

Fig. 8. Cost function of simulation in barrier function based MPC

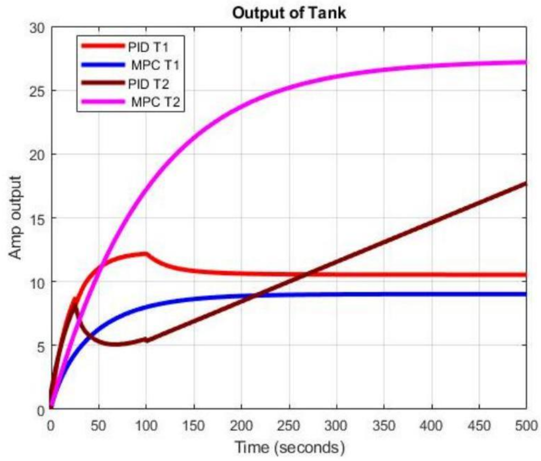

Fig. 9. Comparison between the nonlinear MPC and PID controller; this comparison shows the stability of the system after applying nonlinear MPC and a PID controller in tanks 1 and 2

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