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Analysis of Non-Monotonic Piecewise Lyapunov Stability and Semi-PID Design for Multi-Input Multi-Output Piecewise Affine Systems

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ABSTRACT

This paper presents the Non-monotonic Lyapunov (NML) approach for assessing the stability and stabilization of discrete time piecewise affine dynamical systems. Traditional Lyapunov methods are known for their conservatism, leading to the development of less conservative methods such as NML. Unlike traditional methods, the NML approach does not require strict monotonicity in demonstrating the descent of a Lyapunov functional. In this regard new stability and stabilization criteria based of NML are derived in the form of linear matrix inequalities (LMI) for piecewise affine systems. The NML based method is used to design a semi-PID controller for multi-input multi-output piecewise systems. Also, an optimal semi-PID controller design algorithm is derived in this paper to achieve optimal trajectories and control signal . The effectiveness of this approach is demonstrated in the PID designing for such systems. The paper provides illustrative examples and simulation results to showcase the effectiveness of the NML approach.

1. Introduction

Control and analysis of piecewise affine systems have gained significant attention in recent times due to their practicality in modelling and approximating hybrid and nonlinear systems [1-4]. In this regard the problem of global exponential stability analysis of the origin of continuous-time continuous piecewise affine (CPWA) systems is investigated in [1]. In [1], the stability analysis considers piecewise quadratic (PWQ) Lyapunov functions (LF) and a ramp-based implicit representation of PWA systems. Sufficient convex stability conditions are obtained in the form of a Semidefinite Programming (SDP) problem. Also, researchers introduced the method of piecewise quadratic Lyapunov functions in [2] for analysing the stability of continuous-time Piecewise affine systems. The mentioned researches care about continues-times systems, while some techniques have also been explored for control and analysis of discrete-time piecewise affine systems, as seen in studies such as [3] and [4]. The stability analysis of PWA discrete systems is studied in [3] without considering the conservatism reduction of Lyapunov method. A

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comprehensive analysis and comparison of various stability techniques for discrete-time Piecewise affine systems can be found in [5].

When dealing with piecewise affine systems, a crucial task is to identify a stable admissible region. To achieve this, researchers have developed stability criteria in the form of linear matrix inequalities (LMIs) based on a Lyapunov functional (LF) approach, which is considered an effective method (see [6]). However, most LF-based methods require the LF to be monotonically decreasing, indicating a decrease in the system's initial energy. This strict decrease condition, can lead to conservatism. To address this issue, the non-monotonic Lyapunov (NML) technique can be used. In this approach, the monotonically decreasing condition in LF is replaced with a non-monotonic decreasing condition. Specifically, the non-increasing 1-step difference condition commonly used in stability analysis can be replaced with a non-increasing condition in every m-step difference (see [19]). This allows the NML functional to increase locally for at most (m-1)-steps while maintaining an overall trend of Lyapunov functional in every m-step that is decreasing. The value of m is referred to as the non-monotonicity step. By utilizing the NML technique, conservatism associated with monotonically decreasing LF-based methods can be reduced. This concept was firstly stated as finite-step Lyapunov method, introduced in [8]. Kruszewski also utilized this approach for stabilizing a class of discrete-time Takagi-Sugeno fuzzy models [9]. Later, Ahmadi coined the term "non-monotonic" for this concept and established global asymptotic stability in discrete-time systems by replacing the monotonically decreasing condition with some non-monotonic decreasing ones [10]. Derakhshan and Fatehi introduced a discrete non-monotonic Lyapunov method for analysing the stability of fuzzy control systems by relaxing the monotonicity requirement of Lyapunov's theorem [11]. The non-monotonic Lyapunov function was developed in [12] specifically for discrete-time switching linear systems. This idea was further extended to the N-step ahead Lyapunov function approach, which was used to design a robust $H\infty$ controller for switched systems [13]. Subsequently, the non-monotonic technique was employed to design various controllers, such as optimal controllers [14-15], robust output feedback controllers [16], robust state feedback controllers [17], and robust $H\infty$ controllers for a class of discrete-time nonhomogeneous Markovian jump linear systems [18]. In addition, state feedback controllers for discrete time-delay systems were designed using the non-monotonic technique in [19]. Overall, the non-monotonic Lyapunov method has proven to be a versatile and effective tool for analysing the stability of piecewise affine systems and designing corresponding controllers.

The control of piecewise linear systems is a highly sought-after area of study due to its theoretical complexity and practical applications. Although advanced control theories have been proposed, the proportional-integral-derivative (PID) controller remains the dominant choice in industry due to its simplicity and satisfactory performance for many industrial plants [20]. Most research on PID design techniques focuses on stable single-input-single-output (SISO) processes in continuous time, despite the fact that many industrial processes are multi-input-multi-output (MIMO) and must be implemented in discrete-time. To address these issues, researchers have proposed various approaches. For instance, Lyapunov technique was used to design a PID controller for networked control systems [21], while LQR technique was employed to design a state-space digital PID controller for multivariable analogue systems [22]. Another study presented a graphical tuning method for PI/PID controllers for first order and second order plus time delay systems using the dominant pole placement approach [23]. Additionally, an $H\infty$ controller for continuous systems was designed using a neutral system approach [24].

One important consideration is how to design a PID controller for unstable multi-input multi-output time-varying delay discrete-time systems using a less conservative method that maintains the simplicity of the PID control structure and existing loops in industry, while also ensuring stability and optimality. This paper proposes a non-monotonic piecewise linear Lyapunov (NMPL) approach to investigate the stability of piecewise linear discrete-time systems and offers an optimal PID design technique for unstable MIMO piecewise linear discrete-time systems. Compared to the ordinary piecewise Lyapunov method, the NMPL approach is less conservative, has a larger search space for Lyapunov functional candidate, and allows for choosing non-monotonicity step (m). Section 2 in this paper provides preliminary information, while Section 3 presents the main results, including stability and stabilization theorems. Numerical examples are used to evaluate the proposed method in Section 4, and concluding remarks are given in Section 5.

Nomenclature

R (R^+)	sets of real numbers (positive real numbers)
$\mathcal{R}^{n\times n}$	n-dimensional Euclidean space
I	identity matrix with $n \times n$ dimension
n	number of states in state space representation
l	number of outputs in state space representation
r	number of inputs in state space representation
$0_{n\times m}$	zero matrix with $n \times m$ dimension
$P > 0$ (< 0)	symmetric positive (negative) definite matrix
$\|.\|$	Euclidean vector norm
L	Lipschitz constant

2. Definitions and preliminaries

The purpose of this paper is stability analysis and PID controller design based on non-monotonic Lyapunov (NML) method for piecewise affine discrete time systems. Thus, a NML stability technique is introduced for stability analysis of discrete piecewise affine (DPWA) systems. The main principle in NML stability theorem is that sectionally incremental trend of functional is allowed; but the functional overall trend is decreasing.

Let $x(k) \in R^n$ be the underlying systems information, referred to as the states, and $f: R^n \to R^n$ express how the states change in time and it can be in general nonlinear, non-smooth, or uncertain. Then the discrete time systems can be represented as:

$$x(k+1) = f(x(k)) \tag{1}$$

Then the monotonic Lyapunov stability theorem is represented as follows.

Theorem 1 [10]. The solution $x = 0$ of the system (1) is asymptotically stable, if there exist Lyapunov function $V: R^n \to R$ such that $\forall x(k) \in R^n$:

- $V(0) = 0$
- $V(x(k)) > 0 \ . \ \forall x(k) \neq 0$
- $\|x(k)\| \to \infty \ \Rightarrow \ V(x(k)) \to \infty$
- $V(x(k+1)) - V(x(k)) \leq 0 \quad . \ \forall x \neq 0$ ■

Theorem 2 represents non-monotonic Lyapunov stability theorem.

Theorem 2 [10]**.** Consider dynamical discrete system (1). Then the origin is globally asymptotic stable if there exist non-negative scalers $\gamma_i = 1$ and an unbounded Lyapunov function $V: R^n \to R$ such that:

- $V(0) = 0$
- $V(x(k)) > 0 \ . \ \forall x(k) \neq 0$
- $\|x(k)\| \to \infty \ \Rightarrow \ V(x(k)) \to \infty$
- $\gamma_{m-1}\left(V\big(x(k+m)\big) - V\big(x(k)\big)\right) +$
  $\gamma_{m-2}\left(V\big(x(k+m-1)\big) - V\big(x(k)\big)\right) + \cdots +$
  $\left(V\big(x(k+1)\big) - V\big(x(k)\big)\right) < 0$

Using the Theorems 1 and 2 we are able to present the main results of this study in the next section.

3. Main results

In this paper, the non-monotonic Lyapunov (NML) stability technique is generalized for discrete piecewise affine (DPWA) systems. Eq. (2) represents a DPWA system model.

$$X(k+1) = A_j X(k) \ , \qquad j = 1, \dots, N \tag{2}$$

where A_j are different modes of a linear system. These types of systems are popular. They provide a practical framework for modelling and approximating of hybrid and nonlinear systems. It is well known that Schur stability of A_js is not necessary nor it is sufficient for the overall system to be stable [25]. In this paper, also, stabilization theorems are proposed for multivariable DPWA systems based on generalized NML method. In the stabilization subsection, the problems of PID stabilization and optimal PID control will be presented. Thus, in the following, first NML stability analysis is introduced, then NML based stabilization theorems are proposed for PID designing.

3.1.1. DPWA systems NML stability analysis

As it was mentioned in previous parts, the non-monotonic Lyapunov stability technique lets the Lyapunov functional increases locally in few limited steps, while the overall decreasing property is guaranteed. m-step difference of a functional is defined in (3):

$$\Delta_m V \triangleq V(x_{k+m}) - V(x_k) \tag{3}$$

in which m is the number of steps that the functional can be incremental. The parameter m is called non-monotonicity step. Using Theorem 2 and considering Lyapunov functions $V^i(x_k)$ the m-step non-monotonic stable system will be presented in Theorem 3.

Theorem 3 [10]. Consider dynamical discrete system (1) and considering Lyapunov functions $V^i(x_k)$. Then the origin is globally asymptotic stable if there exist non-negative scalers γ_i and m unbounded Lyapunov functions $V^i(x_k): R^n \to R$ such that:

- $\sum_{i=1}^{m} i V^i(0) = 0$
- $\sum_{i=l}^{m} V^i(0) > 0 \ , \ \ \forall x \neq 0 \ for \ l = 1, \dots, m$
- $\gamma_{m-1}\, \Delta_m V^m + \cdots + \gamma_1\, \Delta_2 V^2 + \Delta_1 V^1 < 0$

Assume a DPWA system as represented in (2). The Lyapunov functional (LF) candidate is considered in the form of (4):

$$V^i(x_k) = X^T(k) P_i X(k) \qquad , \qquad i = 1, \dots, m \tag{4}$$

where the matrices P_i are positive definite. Now, considering the definitions of LF candidate (4), Theorem 4 is presented for stability analysis of DPWA system (2).

Theorem 4. Linear DPWA system (2) with a given initial condition is m-step NML stable if there exist non-negative scalers γ_i and positive definite matrices $P_i \in \mathcal{R}^{n\times n}, i = 1, \dots, m,$ such that:

$$\sum_{i=l}^{m} P_i > 0 \quad , \qquad l = 1, \dots, m \tag{5}$$

$$\sum_{l=1}^{m} \left(\gamma_{l-1}\left[A_{j_l}^T \dots A_{j_2}^T A_{j_1}^T P_l A_{j_1} A_{j_2} \dots A_{j_l} - P_l\right]\right) < 0 \tag{6}$$

$$, \quad j_1, j_2, \dots, j_m = 1, \dots, N$$

Proof: In order to derive stability conditions, first, each term of $\Delta_m V^i$ is calculated in the following steps:

$$\begin{aligned} \Delta_1 V^1 &= X^T(k+1) P_1 X(k+1) - X^T(k) P_1 X(k) \\ \Delta_2 V^2 &= X^T(k+2) P_2 X(k+2) - X^T(k) P_2 X(k) \\ &\vdots \\ \Delta_m V^m &= X^T(k+m) P_m X(k+m) \\ &\quad - X^T(k) P_m X(k) \end{aligned} \tag{7}$$

then using system (2) we have:

$$\begin{aligned} \Delta_1 V^1 &= X^T(k)\left[A_{j_1}^T P_1 A_{j_1} - P_1\right] X(k) \\ \Delta_2 V^2 &= X^T(k)\left[A_{j_2}^T A_{j_1}^T P_2 A_{j_1} A_{j_2} - P_2\right] X(k) \\ &\vdots \\ \Delta_m V^m &= X^T(k)\left[A_{j_m}^T \dots A_{j_2}^T A_{j_1}^T P_m A_{j_1} A_{j_2} \dots A_{j_m} - P_m\right] X(k) \end{aligned}$$

where $A_{j_1} A_{j_2} \dots A_{j_m}$ are m matrices which are multiplied, and $j_1, j_2, \dots, j_m = 1, \dots, N$. Using Theorem 3 and the above, the stability needs to have:

$$\gamma_{m-1}\left[A_{j_m}^T \dots A_{j_2}^T A_{j_1}^T P_m A_{j_1} A_{j_2} \dots A_{j_m} - P_m\right] + \cdots + \gamma_1\left[A_{j_2}^T A_{j_1}^T P_2 A_{j_1} A_{j_2} - P_2\right] + \left[A_{j_1}^T P_1 A_{j_1} - P_1\right] < 0 \quad (8)$$

The inequality (8) can be written in the form of:

$$\sum_{l=1}^{m} \left(\gamma_{l-1}\left[A_{j_l}^T \dots A_{j_2}^T A_{j_1}^T P_l A_{j_1} A_{j_2} \dots A_{j_l} - P_l\right]\right) < 0 \quad (9)$$

$$, \quad j_1, j_2, \dots, j_m = 1, \dots, N$$

should be noted that $\gamma_0 = 1$. This completes the proof.■

Remark 1: In inequality (9) there are m terms as $A_{j_l}^T \dots A_{j_2}^T A_{j_1}^T P_l A_{j_1} A_{j_2} \dots A_{j_l}$. For instance, if $m = 3$, then Eq. (9) will be as:

$$\gamma_2\left[A_{j_3}^T A_{j_2}^T A_{j_1}^T P_3 A_{j_1} A_{j_2} A_{j_3} - P_3\right] + \gamma_1\left[A_{j_2}^T A_{j_1}^T P_2 A_{j_1} A_{j_2} - P_2\right] + \left[A_{j_1}^T P_1 A_{j_1} - P_1\right] < 0 \quad (10)$$

In which $j_1, j_2, j_3 = 1, \dots, N$. Inequality (10) is a set of conditions which should be checked for stability.

Remark 2. In the Theorem 4, *m-step* non-monotonicity leads to complicated and heavy calculations to check stability conditions (5) and (6). But in real application just $m = 2$ can leads to a less conservative and more relaxed Lyapunov stability while lessen the calculations. Corollary 1 introduces *2-step* stability.

Corollary 1. Linear DPWA system (2) with a given initial condition is 2-step NMK stable if there exist non-negative scaler γ_1 and positive definite matrices $P_i \in \mathcal{R}^{n\times n}, i = 1,2$ such that:

$$P_1 > 0 \quad (11a)$$

$$P_1 + P_2 > 0 \quad (11b)$$

$$\gamma_1\left[A_{j_2}^T A_{j_1}^T P_2 A_{j_1} A_{j_2} - P_2\right] + \left[A_{j_1}^T P_1 A_{j_1} - P_1\right] < 0 \quad (11c)$$

$$, \quad \forall\, j_1, j_2 = 1, \dots, N$$

Also, the stability conditions in Corollary 1 can be stated in another form using Schur complement lemma and considering $\gamma_i = 1$. In this regard, Corollary 2 is introduced.

Corollary 2. Linear DPWA system (2) with a given initial condition is 2-step NMK stable if there exist positive definite matrices $P_i \in \mathcal{R}^{n\times n}, i = 1,2$ such that

$$P_1 > 0 \quad (12a)$$

$$P_1 + P_2 > 0 \quad (12b)$$

$$\begin{bmatrix} -(P_1 + P_2) & A_{j_1}^T & A_{j_2}^T A_{j_1}^T \\ * & -P_1^{-1} & 0_{n\times n} \\ * & * & -P_2^{-1} \end{bmatrix} < 0 \quad (12c)$$

where $j_1, j_2 = 1, \dots, N$.

Proof: Considering (11c) with non-negative scaler $\gamma_1 = 1$:

$$\left[A_{j_2}^T A_{j_1}^T P_2 A_{j_1} A_{j_2} - P_2\right] + \left[A_{j_1}^T P_1 A_{j_1} - P_1\right] < 0 \quad (13)$$

Then using Schur complement Lemma:

$$\begin{bmatrix} A_{j_1}^T P_1 A_{j_1} - P_1 - P_2 & A_{j_2}^T A_{j_1}^T \\ A_{j_1} A_{j_2} & -P_2^{-1} \end{bmatrix} < 0 \quad (14)$$

Again, using Schur complement Lemma:

$$\begin{bmatrix} -P_1 - P_2 & A_{j_1}^T & A_{j_2}^T A_{j_1}^T \\ * & -P_1^{-1} & 0_{n\times n} \\ * & * & -P_2^{-1} \end{bmatrix} < 0 \quad (15)$$

The proof is done.

Similarly, the m-step case can be stated as Corollary 3.

Corollary 3. Linear DPWA system (2) with a given initial condition is m-step NMK stable if there exist positive definite matrices $P_i \in \mathcal{R}^{n\times n}, i = 1,2, \dots, m$, such that:

$$\sum_{i=1}^{l} P_i > 0 \quad , \quad l = 1, \dots, m$$

$$\begin{bmatrix} -\varphi_1 & A_{j_1}^T & A_{j_2}^T A_{j_1}^T & \cdots & A_{j_m}^T \dots A_{j_2}^T A_{j_1}^T \\ * & -P_1^{-1} & 0_{n\times n} & \cdots & 0_{n\times n} \\ * & * & -P_2^{-1} & \cdots & 0_{n\times n} \\ \vdots & \vdots & \vdots & \ddots & 0_{n\times n} \\ * & * & * & * & -P_m^{-1} \end{bmatrix} < 0 \quad (16)$$

$$\varphi_1 = (P_1 + P_2 + \cdots P_m)$$

3.2. PID Controller Design

In this section, the problems of designing a stabilizing PID controller and an optimal PID controller will be presented. Many plants in the industry can be properly estimated with pricewise affine linear models, which can be described as the following MIMO discrete state-space representation:

$$\begin{cases} x(k+1) = A_j x(k) + B_j u(k) \\ y(k) = C_j x(k) \end{cases} \quad (17)$$

The matrix $A_j \in \mathcal{R}^{n\times n}$ represents the system dynamics, $B_j \in \mathcal{R}^{n\times r}$ represents the input matrix, and $C_j \in \mathcal{R}^{l\times n}$ represents the output matrix. The aim is to design a MIMO PID controller for the system (17). The discrete PID controller can be formulated as:

$$u(k) = K_p e(k) + K_i \sum_{i=0}^{k-1} e(i) + K_d(y(k) - y(k-1)) \quad (18)$$

where the difference between reference input $r(k)$ and the output $y(k)$ is defined as error:

$$e(k) = r(k) - y(k) \quad (19)$$

and K_p, K_i and K_d have proper dimensions. By applying the controller (18) to the plant (17), the resulting closed-loop control system can be represented as a state-delay system:

$$x(k+1) = A_j x(k) + B_j K_{p_j} e(k) + B_j K_{i_j} \sum_{i=0}^{k-1} e(i) + B_j K_{d_j}((y(k) - y(k-1)) \quad (20)$$

There are different controller gains for each subsystem j of (17). Thus, the controller gains are indexed as K_{p_j}, K_{i_j} and K_{d_j}. Using (19) and (17) in (20), the following is reached.

$$x(k+1) = A_j x(k) + B_j K_{p_j} r(k) - B_j K_{p_j} C_j x(k) + B_j K_{i_j} \sum_{i=0}^{k-1} e(i) + B_j K_{d_j} C_j x(k) - B_j K_{d_j} y(k-1) \quad (21)$$

We try to reformulate the control problem as output feedback. In other word, PID controller designing problem is changed into an output feedback control of an augmented system. For this purpose, new state vector is defined as follows:

$$\bar{X}^T(k) = \left[x^T(k) \quad \sum_{i=0}^{k-1} e^T(i) \quad y^T(k-1) \right] \quad (22)$$

Considering (22) and using a variable change technique, Eq. (21) will be reformulated as follows:

$$\bar{X}(k+1) = \begin{bmatrix} \varphi_2 & B_j K_{i_j} & -B_j K_{d_j} \\ -C_j & I & 0 \\ C_j & 0 & 0 \end{bmatrix} \bar{X}(k) + \begin{bmatrix} B_j K_{p_j} \\ I \\ 0 \end{bmatrix} r(k) \quad (23)$$

$$\textit{where } \varphi_2 = A_j - B_j K_{p_j} C_j + B_j K_{d_j} C_j$$

Also, the output can be rewritten as (24) using new state vector:

$$y(k) = [C_j \quad 0 \quad 0] \bar{X}(k) \quad (24)$$

Finally, the closed loop augmented systems will be as follows:

$$\begin{cases} \bar{X}(k+1) = \tilde{A}_j \bar{X}(k) + \tilde{B}_j r(k) \\ y(k) = \tilde{C}_j \bar{X}(k) \end{cases} \quad (25)$$

where

$$\tilde{A}_j = \begin{bmatrix} \varphi_2 & B_j K_{i_j} & -B_j K_{d_j} \\ -C_j & I & 0 \\ C_j & 0 & 0 \end{bmatrix}$$

$$\tilde{C}_j = [C_j \quad 0 \quad 0], \; \tilde{B}_j = \begin{bmatrix} B_j K_{p_j} \\ I \\ 0 \end{bmatrix} \quad (26)$$

$$\varphi_2 = A_j - B_j K_{p_j} C_j + B_j K_{d_j} C_j$$

In the next two subsections, the stability of the augmented system (25) is investigated in the presence of the controller gains K_{p_j}, K_{i_j} and K_{d_j}. Then, by calculating these controller gains such that the system (25) is stable, the stabilizing controller is reached.

3.2.1. PID stabilization

The NML stability condition for pricewise affine systems given in Section 3.1 can be applied to design a PID controller for a closed loop system (25).

The problem of stabilization is that the closed loop form should uses $\tilde{A}_j$, which contains the controller gains. In order to reduce the complexity, the non-monotonic approach can be simplified and reformulated. In this regard the following Theorem 5 is introduced

Theorem 5. The closed-loop time-delay system (25) is globally asymptotically m-step non-monotonic stabilizable if there exist controller gains K_{p_j}, K_{i_j}, K_{d_j} and positive definite matrices P_k and ψ_k, $(k = 1, 2, \dots, m)$ with appropriate dimension, such that

$$\begin{bmatrix} -P_2 & \tilde{A}_{j_1}^T \\ \tilde{A}_{j_1} & -\psi_1 \end{bmatrix} < 0 \\ \begin{bmatrix} -P_3 & \tilde{A}_{j_2}^T \\ \tilde{A}_{j_2} & -\psi_2 \end{bmatrix} < 0 \\ \begin{bmatrix} -P_4 & \tilde{A}_{j_3}^T \\ \tilde{A}_3 & -\psi_3 \end{bmatrix} < 0 \\ \vdots \\ \begin{bmatrix} -P_1 & \tilde{A}_{j_m}^T \\ \tilde{A}_{j_m} & -\psi_m \end{bmatrix} < 0 \quad (27)$$

where $j_1, j_2, \dots, j_m = 1, \dots, N$.

while minimizing $\{P_1\psi_1 + P_2\psi_2 + \dots + P_m\psi_m\}$,

Subject to:

$$\begin{bmatrix} P_1 & I \\ I & \psi_1 \end{bmatrix} \geq 0, \quad \begin{bmatrix} P_2 & I \\ I & \psi_2 \end{bmatrix} \geq 0, \dots, \quad \begin{bmatrix} P_m & I \\ I & \psi_m \end{bmatrix} \geq 0$$

Proof: As it was stated in Theorem 2 the stability condition was $\gamma_{m-1}\,\Delta_m V^m + \dots + \gamma_1\,\Delta_2 V^2 + \Delta_1 V^1 < 0$. But in many applications, that's enough to let the Lyapunov functional candidate to increase in *m-step* and overlay decreasing. Thus, we consider $\Delta_m V^m < 0$ for stability. Then:

$$\begin{cases} x(k+1) = A_j x(k) + B_j u(k) \\ y(k) = C_j x(k) \end{cases} \quad (28)$$

by adding and subtracting the term $A_{j_m}^T\, P_m\, A_{j_m}$ to (28):

$$\Delta_m V^m = X^T(k) \left[A_{j_m}^T \dots A_{j_2}^T A_{j_1}^T\, P_1\, A_{j_1} A_{j_2} \dots A_{j_m} - P_1 - A_{j_m}^T\, P_m\, A_{j_m} + A_{j_m}^T\, P_m\, A_{j_m} \right] X(k) < 0 \quad (29)$$

where the parameter P_m is positive definite.

Then, (29) can be rewritten in form of (30a). Then (30a) can be shown in the form of inequalities (30b) and (30c) in two steps by some manipulation. Then, by defining $y_m(k) = A_{j_m} X(k)$ in (30c), (31) is derived.

$$\Delta_m V^m = y_m^T(k)\left[A_{j_{m-1}}^T \dots A_{j_2}^T A_{j_1}^T\, P_1\, A_{j_1} A_{j_2} \dots A_{j_{m-1}} - P_m\right] y_m(k) + X^T(k)\left[A_{j_m}^T\, P_m\, A_{j_m} - P_1\right] X(k) < 0 \tag{31}$$

If the same procedure be repeated m time, then we will have:

$$\Delta_m V^m = y_2^T(k)\left[A_{j_1}^T\, P_1\, A_{j_1} - P_2\right] y_2(k) + \cdots + y_m^T(k)\left[A_{j_{m-1}}^T\, P_{m-1}\, A_{j_{m-1}} - P_m\right] y_m(k) + X^T(k)\left[A_{j_m}^T\, P_m\, A_{j_m} - P_1\right] X(k) < 0 \tag{32}$$

Obviously, inequality (32) holds and the system will be stable if all the following inequalities holds:

$$\left(A_{j_1}^T\, P_1\, A_{j_1} - P_2\right) < 0 \tag{33.1}$$

$$\left(A_{j_2}^T\, P_2\, A_{j_2} - P_3\right) < 0 \tag{33.2}$$

$$\vdots \qquad\qquad \vdots$$

$$\left(A_{j_{m-1}}^T\, P_{m-1}\, A_{j_{m-1}} - P_m\right) < 0 \tag{33.(m-1)}$$

$$\left(A_{j_m}^T\, P_m\, A_{j_m} - P_1\right) < 0 \tag{33.m}$$

Now inequalities (33) can be rewritten in the form of (34) using Schure complement Lemma:

$$\begin{bmatrix} -P_2 & A_{j_1}^T \\ A_{j_1} & -P_1^{-1} \end{bmatrix} < 0 \quad \begin{bmatrix} -P_3 & A_{j_2}^T \\ A_{j_2} & -P_2^{-1} \end{bmatrix} < 0 \quad \vdots \quad \begin{bmatrix} -P_1 & A_{j_m}^T \\ A_{j_m} & -P_m^{-1} \end{bmatrix} < 0 \tag{34}$$

The inequalities (34) are non-convex because of P_k^{-1} terms. These non-convex criteria can convert to convex LMI problem using Cone Complementarity Problem introduced in [26], which can be solved using a repetitive algorithm introduced in [27]. In this regard the positive definite matrices ψ_k, $(k = 1,2, \dots m)$ should exists and the following LMIs should hold:

$$\begin{bmatrix} -P_2 & A_{j_1}^T \\ A_{j_1} & -\psi_1 \end{bmatrix} < 0 \quad \begin{bmatrix} -P_3 & A_{j_2}^T \\ A_{j_2} & -\psi_2 \end{bmatrix} < 0 \quad \vdots \quad \begin{bmatrix} -P_1 & A_{j_m}^T \\ A_{j_m} & -\psi_m \end{bmatrix} < 0 \tag{35}$$

$$\Delta_m V^m = X^T(k)\left[A_{j_m}^T \dots A_{j_2}^T A_{j_1}^T\, P_1\, A_{j_1} A_{j_2} \dots A_{j_m} - P_1 - A_{j_m}^T\, P_m\, A_{j_m} + A_{j_m}^T\, P_m\, A_{j_m}\right] X(k) < 0 \tag{30a}$$

$$\Delta_m V^m = X^T(k)\left[A_{j_m}^T \left(A_{j_{m-1}}^T \dots A_{j_2}^T A_{j_1}^T\, P_1\, A_{j_1} A_{j_2} \dots A_{j_{m-1}} - P_m\right) A_{j_m}\right] X(k) + X^T(k)\left[\left(A_{j_m}^T\, P_2\, A_{j_m} - P_1\right)\right] X(k) < 0 \tag{30b}$$

$$\Delta_m V^m = X^T(k) A_{j_m}^T \left[\left(A_{j_{m-1}}^T \dots A_{j_2}^T A_{j_1}^T\, P_1\, A_{j_1} A_{j_2} \dots A_{j_{m-1}} - P_m\right)\right] A_{j_m} X(k) + X^T(k)\left[\left(A_{j_m}^T\, P_m\, A_{j_m} - P_1\right)\right] X(k) < 0 \tag{30c}$$

while minimizing $\{P_1\psi_1 + P_2\psi_2 + \cdots + P_m\psi_m\}$,
Subject to:

$$\begin{bmatrix} P_1 & I \\ I & \psi_1 \end{bmatrix} \geq 0, \quad \begin{bmatrix} P_2 & I \\ I & \psi_2 \end{bmatrix} \geq 0, \dots, \quad \begin{bmatrix} P_m & I \\ I & \psi_m \end{bmatrix} \geq 0$$

The closed loop augmented system (25) is considered. For stability analysis of closed loop augmented system (PID designing procedure), the matrices A_j in (35) are replaced with closed loop matrix coefficient $\tilde{A}_j$. Then inequality (27) is reached and the proof is completed.

Remark 3. The obtained controller in Theorem 5 can stabilize both stable and unstable systems. The plant can be Multi-Input Multi-Output (MIMO).

Remark 4. By Choosing bigger m, the computations will increase as the conservatism decreases. Choosing m is a trade-off between heavy computations and reducing the conservatism. In most of the applications it may be enough to choose $m = 2$.

3.2.2. Optimal PID control

In subsection 3.2.1, the stability is analysed for the closed-loop system (25). But the obtained controller may be non-optimal. In this regard, a performance index is introduced in this section and the aim is to minimize it to reach optimal performance. Assume the following cost function for the system (25):

$$J = \|\bar{X}(k)\|_R^2 + \|\hat{u}(k)\|_F^2 \tag{36}$$

where $\|\bar{X}(k)\|_R^2 = \sum_{k=0}^{\infty} \bar{X}^T(k) R \bar{X}(k)$, $\|\hat{u}(k)\|_F^2 = \sum_{k=0}^{\infty} \hat{u}^T(k) F \hat{u}(k)$ and $R \coloneqq \bar{R}[0 \quad I \quad 0]$, $\bar{R}$ and F are weighting factors.

Assuming the control law as (18), using the state space representation (17), Eq. (19) and new state vector defined in (22), we can rewrite control law as follows:

$$\hat{u}(k) = K_{p_j} e(k) + K_{i_j} \sum_{i=0}^{k-1} e(i) + K_{d_j}\big(y(k) - y(k-1)\big) = K_j \bar{E} \bar{X}(k) + K_j \bar{D} r(k) \tag{37}$$

in which $\bar{E}=\begin{bmatrix}-C_j & 0 & 0\\ 0 & I & 0\\ -C_j & 0 & I\end{bmatrix}$, $\bar{D}=\begin{bmatrix}I\\0\\0\end{bmatrix}$ and $K_j=\begin{bmatrix}K_{p_j} & K_{i_j} & K_{d_j}\end{bmatrix}$.

where $j=j_1,j_2,\dots,j_m=1,\dots,N$. Considering the state-space representation (25) with zero initial conditions and $r(k)=0$, the objective function for the stability of the system in the presence of PID controller and optimal performance of the controller is assumed as (38):

$$\bar{X}^T(k)R\bar{X}(k)+\hat{u}^T(k)F\hat{u}(k)\le-\Delta_m V(k) \tag{38}$$

summing both sides of the inequality (38) from 0 to ∞, we have:

$$\sum_{k=0}^{\infty}\bar{X}^T(k)R\bar{X}(k)+\sum_{k=0}^{\infty}\hat{u}^T(k)F\hat{u}(k)\le V(x(0))-V(x(\infty)) \tag{39}$$

Since the stability of the system has been provided, it follows that $V(x(k))\to 0$ as $k\to\infty$; Hence:

$$J\le V(x(0)) \tag{40}$$

which implies that the upper bound of the cost function depends on the initial condition. Considering (4) we have:

$$J\le X^T(0)\,P_j\,X(0) \tag{41}$$

Suppose that the initial state of the system is arbitrary but belongs to the set $S=\{x(0)\in R^n\colon\ x(0)=UM,\ M^TM\le 1\}$; where U is a given matrix. Thus considering Eq. (22), the cost bound then leads to

$$\begin{aligned}J&\le\begin{bmatrix}x(0)\\e(-1)\\y(-1)\end{bmatrix}^T P_j\begin{bmatrix}x(0)\\e(-1)\\y(-1)\end{bmatrix}=M^TU^TP_jUM\\&\le\lambda_{max}\left(U^TP_jU\right)M^TM\\&\le\lambda_{max}\left(U^TP_jU\right)\end{aligned} \tag{42}$$

where $\lambda_{max}(.)$ denotes the maximum eigenvalue of the matrix $(.)$. Also, it is assumed that $e(-1)=0$ and $y(-1)=0$. Then the upper bound of the cost function will be as:

$$J<\lambda_{max}\left(U^TP_jU\right) \tag{43}$$

The obtained inequality (43) is the upper bound of cost function which depends on the parameter P_j. This obtained condition should be considered in calculations. The following theorem gives optimal NML based stabilization conditions.

Theorem 6. The closed-loop time-delay system (25) with $r(k)=0$ is optimally globally asymptotically m-step non-monotonic stabilizable if there exist controller gains K_p, K_i, K_d, and positive definite matrices P_k, $\psi_k\in R^n$, $(k=1,2,\dots,m)$, and any matrix such that:

$$\begin{bmatrix}-P_2 & \tilde{A}_{j_1}^T\\ \tilde{A}_{j_1} & -\psi_1\end{bmatrix}<0 \tag{44.1}$$

$$\begin{bmatrix}-P_3 & \tilde{A}_{j_2}^T\\ \tilde{A}_{j_2} & -\psi_2\end{bmatrix}<0 \tag{44.2}$$

$$\begin{bmatrix}-P_4 & \tilde{A}_{j_3}^T\\ \tilde{A}_{j_3} & -\psi_3\end{bmatrix}<0 \tag{44.3}$$

⋮ ⋮

$$\begin{bmatrix}-P_1+R & K_{j_m}^T\bar{E}^T & \tilde{A}_{j_m}^T\\ * & -F^{-1} & 0_{n\times n}\\ * & * & -\psi_m\end{bmatrix}<0 \tag{44.m}$$

where $j_1,j_2,\dots,j_m=1,\dots,N$.

while minimizing $\{P_1\psi_1+P_2\psi_2+\cdots+P_m\psi_m\}$,

Subject to: $\begin{bmatrix}P_1 & I\\ I & \psi_1\end{bmatrix}\ge 0$, $\begin{bmatrix}P_2 & I\\ I & \psi_2\end{bmatrix}\ge 0$, …, $\begin{bmatrix}P_m & I\\ I & \psi_m\end{bmatrix}\ge 0$

furthermore, the closed-loop cost function Eq. (43) should satisfies; then the output feedback control law Eq. (37) is a guaranteed cost control law.

***Proof*:** Considering the closed loop system (25), the designing problem of PID controller reduces to the design of a stable state feedback controller as given in Theorem 5 including the extra optimality condition (38). To establish the optimal controller for the closed-loop PID control system, consider the following objective function:

$$\bar{X}^T(k)R\bar{X}(k)+\hat{u}^T(k)F\hat{u}(k)\le-\Delta_m V(k) \tag{45}$$

Considering (37) with $r(k)=0$, the Eq. (45) can be written in the following form:

$$\bar{X}^T(k)R\bar{X}(k)+\left(K_{j_m}\bar{E}\bar{X}(k)\right)^T F\left(K_{j_m}\bar{E}\bar{X}(k)\right)<-\Delta_m V(k) \tag{46}$$

in which $R:=\bar{R}\,[0\quad I\quad 0]$, where $\bar{R}$ and F are weighting factors selected by the designer. Then using (28):

$$\begin{aligned}&\bar{X}^T(k)\left[\tilde{A}_{j_m}^T\dots\tilde{A}_{j_2}^T\tilde{A}_{j_1}^T\,P_1\,\tilde{A}_{j_1}\tilde{A}_{j_2}\dots\tilde{A}_{j_m}-P_1\right]\bar{X}(k)\\&+\bar{X}^T(k)R\bar{X}(k)\\&+\left(K_{j_m}\bar{E}\bar{X}(k)\right)^T F\left(K_{j_m}\bar{E}\bar{X}(k)\right)<0\end{aligned} \tag{47}$$

which can be written in the form of:

$$\bar{X}^T(k)\left[\tilde{A}_{j_m}^T\dots\tilde{A}_{j_2}^T\tilde{A}_{j_1}^T\,P_1\,\tilde{A}_{j_1}\tilde{A}_{j_2}\dots\tilde{A}_{j_m}-P_1+R+\bar{E}^T\,K_{j_m}^T F\,K_{j_m}\bar{E}\right]\bar{X}(k)<0 \tag{48}$$

In order to hold (48), it is sufficient to have $\tilde{A}_{j_m}^T\dots\tilde{A}_{j_2}^T\tilde{A}_{j_1}^TP_1\tilde{A}_{j_1}\tilde{A}_{j_2}\dots\tilde{A}_{j_m}-P_1+R+\bar{E}^TK_{j_m}^TFK_{j_m}\bar{E}<0$. Then, in a similar approach to proof of Theorem 5, and using Schur complement lemma:

$$\begin{bmatrix} -P_2 & \tilde{A}_{j_1}^T \\ \tilde{A}_{j_1} & -P_1^{-1} \end{bmatrix} < 0 \quad (49.1)$$

$$\begin{bmatrix} -P_3 & \tilde{A}_{j_2}^T \\ \tilde{A}_{j_2} & -P_2^{-1} \end{bmatrix} < 0 \quad (49.2)$$

$$\begin{bmatrix} -P_4 & \tilde{A}_{j_3}^T \\ \tilde{A}_{j_3} & -P_3^{-1} \end{bmatrix} < 0 \quad (49.3)$$

$$\vdots \qquad \vdots$$

$$\begin{bmatrix} -P_1 + R & \bar{E}^T K_{j_m}^T & \tilde{A}_{j_m}^T \\ * & -F^{-1} & 0_{n\times n} \\ * & * & -P_m^{-1} \end{bmatrix} < 0 \quad (49.m)$$

The inequalities (49) are non-convex because of P_k^{-1} terms. Using Cone Complementarity Problem introduced in [26] and the same as Theorem 5, (44) can be reached. In this regard the following LMIs should hold:

$$\begin{bmatrix} -P_2 & \tilde{A}_{j_1}^T \\ \tilde{A}_{j_1} & -\psi_1 \end{bmatrix} < 0 \quad (50.1)$$

$$\begin{bmatrix} -P_3 & \tilde{A}_{j_2}^T \\ \tilde{A}_{j_2} & -\psi_2 \end{bmatrix} < 0 \quad (50.2)$$

$$\begin{bmatrix} -P_4 & \tilde{A}_{j_3}^T \\ \tilde{A}_{j_3} & -\psi_3 \end{bmatrix} < 0 \quad (50.3)$$

$$\vdots \qquad \vdots$$

$$\begin{bmatrix} -P_1 + R & K_{j_m}^T \bar{E}^T & \tilde{A}_{j_m}^T \\ * & -F^{-1} & 0_{n\times n} \\ * & * & -\psi_m \end{bmatrix} < 0 \quad (50.m)$$

where $j_1, j_2, \dots, j_m = 1, \dots, N$.
while minimizing $\{P_1\psi_1 + P_2\psi_2 + \cdots + P_m\psi_m\}$,
Subject to:

$$\begin{bmatrix} P_1 & I \\ I & \psi_1 \end{bmatrix} \geq 0$$
$$\begin{bmatrix} P_2 & I \\ I & \psi_2 \end{bmatrix} \geq 0$$
$$\begin{bmatrix} P_3 & I \\ I & \psi_3 \end{bmatrix} \geq 0 \quad (51)$$
$$\vdots$$
$$\begin{bmatrix} P_m & I \\ I & \psi_m \end{bmatrix} \geq 0$$

If matrix inequalities (44.1) to (44.m) have feasible solution P_k's , ψ_k and $K_{j_m} = [K_{p_{j_m}} \quad K_{i_{j_m}} \quad K_{d_{j_m}}]$, and the corresponding closed-loop cost function (43) satisfies; then the output feedback control law is a guaranteed cost control law.

4. Numerical example

The results presented in this section were generated using MATLAB (version 2023a) on a computer equipped with an Intel(R) Core (TM) i5 CPU M430 running at 2.27 GHz and 4 GB of RAM, operating on Windows 10. The YALMIP toolbox, along with the SeDuMi and SDPT3 solvers, were utilized for the computations. It should be emphasized that minor differences in outcomes may arise by adjusting the settings and selecting different solvers.

Example 1. The following matrices defining dynamical model of a linear system:

$$A_1 = \begin{bmatrix} 0.4 & 0 \\ -\sqrt{7/8} & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & \sqrt{7/8} \\ 0 & 0.4 \end{bmatrix} \quad (52)$$

The system's stability can't be demonstrated using the conventional Lyapunov approaches, which was verified using the MATLAB LMI toolbox. Conversely, the introduction of the non-monotonic Lyapunov method (Theorem 4) ensures stability, leading to the attainment of following positive definite matrices.

$$P_1 = \begin{bmatrix} 90.87 & 0 \\ 0 & 90.87 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 67.6 & 0 \\ 0 & 67.6 \end{bmatrix}$$

In this example, the non-monotonicity step is set at $m = 2$. It is evident that the non-monotonic based approach introduced here is less conservative.

Example 2. Consider system (17) with the following modes:

$$A_1 = \begin{bmatrix} 1 & 0.1 \\ -0.5 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_1 = [0 \quad 1]$$
$$A_2 = \begin{bmatrix} 0.5 & 0.6 \\ -0.6 & 0.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_2 = [1 \quad 0] \quad (53)$$

For the two components defined in (53), conventional Lyapunov approaches can't guarantee the stability of closed loop system. But using NML based method in Theorem 5 provides an output feedback semi PID controller which stabilizes this system. Therefore, assuming non-monotonicity step $m = 2$, the BMIs (27) have feasible solutions:

$$P_1 = \begin{bmatrix} 2144.7 & 291.1 & -772.6 & -81.1 \\ 291.1 & 1492.2 & -795 & -129.2 \\ -772.6 & 795 & 769.5 & 79.2 \\ -81.1 & 129.2 & 79.2 & 75.6 \end{bmatrix}$$
$$P_2 = \begin{bmatrix} 2151.4 & 295.1 & -773.9 & -77.9 \\ 295.1 & 1490.8 & -794.9 & -130.4 \\ -773.9 & -749.9 & 768.3 & 79.8 \\ -77.9 & -130.4 & 79.8 & 74.2 \end{bmatrix}$$

And the controller gains are as follows:

$$K_1 = [1.4232 \quad 0.7281 \quad -0.0288]$$
$$K_2 = [0.7053 \quad 0.6 \quad -0.059]$$

The following represents non-monotonic decreasing of Lyapunov Function. Is can be seen that Lyapunov

Function increase locally in some samples, but it is overall decreasing. As it is seen in Fig. 1, increasing steps are less than 2 steps, because non-monotonicity step is $m = 2$ and the Lyapunov Function is allowed to increase up to 2 steps.

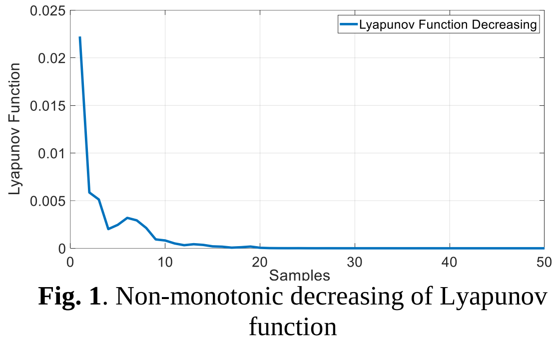


Fig. 1. Non-monotonic decreasing of Lyapunov function

The following show state trajectories form initial condition $x_0 = \begin{bmatrix}1\\1\end{bmatrix}$ and $u(t) = 0$.

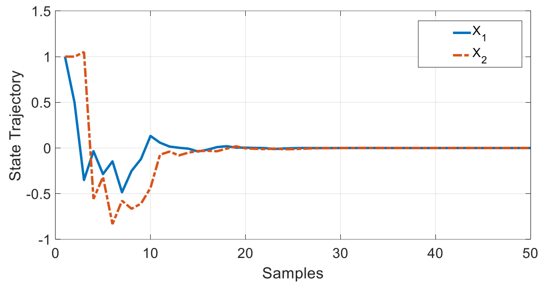


Fig. 2. State Trajectories of first component of (53)

Switching pattern between two subsystems in (52) is considered randomly and as follows:

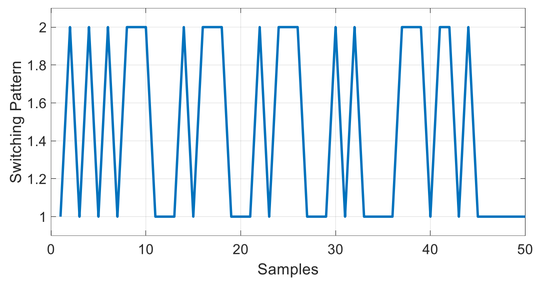
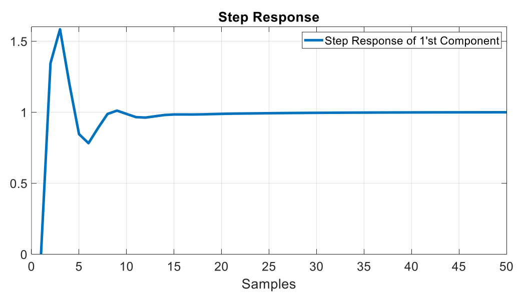
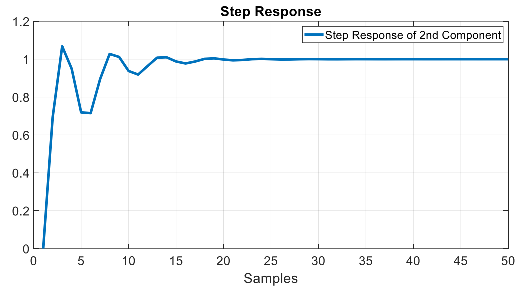


Fig. 3. Switching pattern between subsystems

The step responses of the system are illustrated in Fig.4. It shows that designed semi-PID controller stabilize the system.



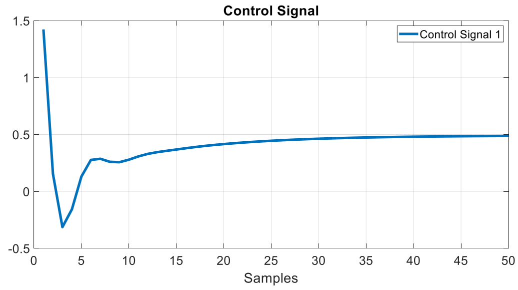
(a). Step response of first component



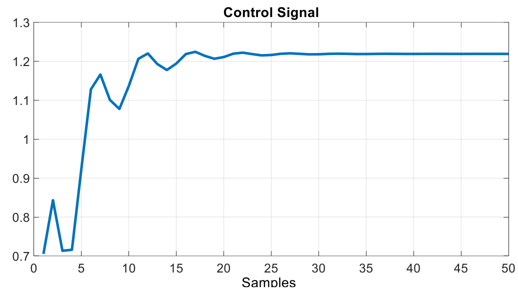
(b). Step response of Second component

Fig. 4. step response of components of the output of system (53)

The Fig. 5 represents control signal of each mode in system (53).



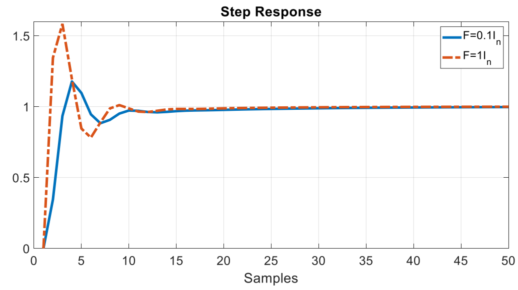
(a)



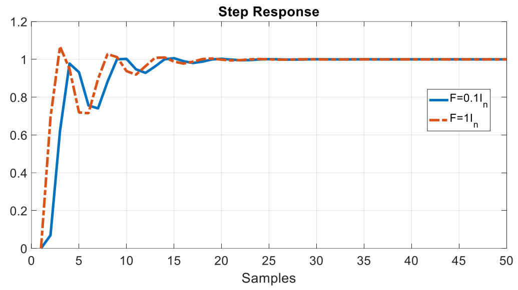
(b)

Fig. 5. Control Signal of system (53). (a): 1'st Component, (b): 2nd component

Example 3. An optimal PID controller is designed using Theorem 6 for the system (53). Let the, the weighting matrices $\bar{R} = I_n$ and $F = 0.1I_n$. Therefore, NML based optimal PID controller can reduce the overshoot in step response as the state trajectories are limited in cost function.



(a) Step response of first component



(b) Step response of 2nd component

Fig. 6. Optimal PID controller design for (53) with same $\bar{R} = I_n$

Fig. 6 compares the changing in the values for F with similar R. When R keeps the same, a larger F punishes the control signal, thus, the control signal will be softer. However, larger F leads to greater tracking error. The optimal semi-PID controller can be designed for unstable multi-input multi-output systems using Theorem 6 and weighting matrices R and F can be selected by control designer in appropriate way to have suitable responses.

5. Conclusion

In this paper, we introduced the Non-monotonic Lyapunov (NML) approach as a novel method for evaluating the stability and stabilization of discrete-time piecewise affine dynamical systems. The NML approach offers a less conservative alternative to traditional Lyapunov methods, allowing for more flexibility in demonstrating Lyapunov functional descent without strict monotonicity requirements. By deriving new stability and stabilization criteria in the form of linear matrix inequalities (LMI) tailored for piecewise affine systems, we have shown the applicability and effectiveness of the NML approach. Moreover, we utilized the NML-based method to design a semi-Proportional-Integral-Derivative (PID) controller for multi-input multi-output piecewise systems, showcasing its practical utility in control system design. Through illustrative examples and simulation results, we have demonstrated the efficacy and advantages of employing the NML approach in PID controller design which shows that overall, the NML approach presents a promising avenue.

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