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Fixed-Time Containment Tracking Control of Fractional-Order Multi-Vehicle Systems with Multiple Leaders: SMC approach

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ABSTRACT

This paper addresses sliding mode control (SMC) design for disturbed fractional-order multi-vehicle networks in order to achieve containment tracking within a certain settling time. The multi-leader case is investigated where the aim of the containment protocol design is that the states of the fractional-order followers eventually are placed inside a convex hull made by the states of the leaders. The convergence rate is designed such that achieving the containment tracking occurs in a fixed-time manner. Unlike the previous works on finite-time containment control protocols of multi-agent systems, here, we offer a tractable design as the upper limit of the settling time of the convergence is achieved independent of the preliminary conditions of the vehicles' states. A novel SMC approach is proposed which enables the multi-vehicle network to reach the containment tracking at presence of the external disturbances. The numerical simulations reveal the correctness and effectiveness of the proposed theoretical approaches.

1. Introduction

Multi-vehicle systems (MVSs) consisting of a number of vehicles (moving agents) that communicate with one another so as to reach the desired agreement on states or outputs, have absorbed abundant interest in various areas, for instance control of unmanned ground and aerial robots [1, 2], multi-satellite systems [3], cooperative dynamical games [4], aerial refueling and etc. In cooperative control of multi-vehicle systems, most of the results for all of the vehicles have considered integer-order dynamics [5-8]. However, vehicles with fractional-order dynamics collaborating with each other, have opened a new research topic in cooperative control of multi-vehicle systems. In general, fractional-order systems which are capable of characterizing complicated dynamics such as lossy transmission, viscoelasticity, heat conduction process, etc. have been under investigation during the last decade [9, 10]. The key and important features of fractional order models are memory description and the non-local behavior [11]. When multiple vehicles such as drones, ground mobile robots, or underwater robots are placed in fully or partially viscous environments, they cannot be described by ordinary integer-order dynamics such as single or double integrators. Such multiple robotic systems which cooperate in viscous fluids can only be characterized with fractional-order dynamics. Viscosity of the fluid determines the fractional order of the dynamics of the vehicles [12, 13]. In the movement of the

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robots in viscous areas, the leaders which are mostly equipped with high-quality sensors make a formation of convex hull, inside which the followers are placed. As the formation of the leaders move, the followers which are inside the convex hull as the safe area move along with the leaders provided that the containment protocol works well. Different scenarios of cooperative control have been implemented on fractional-order agents [14, 15]. In [16], the inverse-optimal consensus control were investigated in single-integrator multi-vehicle systems with Caputo and Grunwald-Letnikov fractional orders. In [17], a distributed event-based adaptive fuzzy consensus protocol was proposed for a class of nonlinear fractional-order multi-vehicle systems. One of the challenging problems in cooperative control of is the multi-leader problem which is called containment tracking which has intrigued many researchers. Its main expected goal is that all the followers can reach the final position within the convex hull spanned by two or more leaders [18-20]. For example, in [21] necessary and sufficient conditions have been proposed for containment control under directed topologies with dynamic leaders. In [22], the containment tracking control of linear multi-vehicle systems with stochastic perturbations has been investigated where the convex hull formed by the leaders is stochastic and convergence to this hull has been guaranteed by the proposed algorithm. Additionally, in [23], a fixed-time containment tracking protocol has been suggested for multi-dimensional nonlinear multi-vehicle systems. In [24], the containment of fractional-order multi-vehicle systems with a directed communication topology was studied. In addition, [25] has discussed the output feedback fixed-time containment for nonlinear multi-vehicle systems with switching graphs under unknown leader dynamics. In [26], observer-based containment control was investigated for an uncertain nonlinear multi-vehicle systems through active disturbance rejection control and back-stepping approach. Other works such as [27] have considered practical implementation issues such as input saturation in the containment tracking of MVSs with unknown leader inputs. In [28], the formation of multi-vehicle systems with fractional order dynamics has been investigated by using the fixed-time Lyapunov stability theorem. It is suggested that the fixed-time formation tracking is achievable within a certain settling time.

In the previous works, the results have been derived considering the fact that the dynamics are integer order integrators. In our paper, we consider fractional-order dynamics whose theory is different and can cover more advanced applications. This approach is a more general approach where the results can be reduced to integer-order dynamics. Thus, integer-order multi-agent systems can then be considered as a very special case of fractional-order multi agent systems.

Additionally, previous papers have investigated the consensus of agents with the presence of one leader, while in our paper the convergence of agents with a specific formation with multiple leaders has been studied. We have also developed a novel SMC approach in order to design a suitable protocol to achieve containment and compensate for the external disturbances at the same time. The novelties and contributions of this paper are outlined below:

- Unlike most previous containment protocol methods such as [25-27] which work for integer-order vehicles, the proposed approach in this paper studies fractional-order MVSs to design an SMC-based containment protocol.
- On the other hand, asymptotic convergence of cooperative control protocols such as [29, 30] cannot necessarily work well in practice, since they guarantee consensus and containment in infinite time rather than finite time. Even finite-time protocols suffer from an important drawback. The settling time in finite time approaches depends on the initial conditions of the vehicles' states. In myriad of cases, being aware of the preliminary conditions of the vehicles is not possible or not easily obtained, though. This encourages us to design a fixed-time approach where it is ensured that the settling time has an upper limit independent of the initial conditions. This approach provides the designers with a method, by following which, the desirable control performances are obtained at a certain time, which is independent of the preliminary conditions.

2. Definitions

The interactions among the vehicles in MVSs are described by graphs. Consider a group of $N+M$ vehicles including N followers and M leaders. The followers are labeled from 1 to N while $N+1$ through $N+M$ are the labels of the leaders. $\boldsymbol{\mathcal{F}}=\{1,2,\cdots,N\}$ contains the indices of the followers and $\boldsymbol{\mathcal{L}}=\{N+1,N+2,\cdots,N+M\}$ includes those of the leaders. Suppose $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ is a graph that describes the interaction among the vehicles with $\mathcal{V}=\{v_1,v_2,v_3,\cdots,v_{N+M}\}$ being the set of $N+M$ nodes of $\mathcal{G}$ and $\mathcal{E}=\{\varepsilon_{ij}=(v_i,v_j)\}\subseteq\mathcal{V}\times\mathcal{V}$ the set of its edges. The set of neighbors for vehicle i is specified by $N_i=\{j:(v_i,v_j)\in\mathcal{V},\ i\neq j\}$. The adjacency matrix $\mathcal{A}=[a_{ij}]\in\mathbb{R}^{(N+M)\times(N+M)}$ is defined for the graph $\mathcal{G}$ with non-negative elements a_{ij} where $a_{ij}>0$ if $\varepsilon_{ij}\in\mathcal{E}$, and $a_{ij}=0$ if $\varepsilon_{ij}\notin\mathcal{E}$. The Laplacian matrix of $\mathcal{G}$ is introduced as $L=[l_{ij}]\in\mathbb{R}^{(N+M)\times(N+M)}$ with $l_{ij}=-a_{ij}$ for $i\neq j$ and $l_{ii}=\sum_{j=1,j\neq i}^{N}a_{ij}$.

Definition 1 [31]. The definition of Reimann-Liouville fractional integral and derivative is given as follows:

$$D^{\rho}f(t)=\begin{cases}\frac{1}{\Gamma(-\rho)}\int_a^t(t-\tau)^{-\rho-1}f(\tau)d(\tau), & \rho<0\\ f(t) & \rho=0\\ D^n[{}_aD_t{}^{\rho-n}f(t)], & \rho>0\end{cases},$$

with $\Gamma(.)$ the Gamma function and n specifying the first integer larger than ρ. $D^{\rho}f(t)$ indicates the derivative operator for $\rho>0$, and the fractional integral for $\rho<0$. For $\rho=1$, the fractional derivative is reduced to time derivative of order 1. Similarly, for other integer numbers as $\rho=n$, higher order time-derivatives are obtained.

Lemma 1 [32]. The following inequalities are true for non-negative real numbers in the form of $y_1,y_2,\ldots,y_N$:
1. if $0<k<1$, then $\sum_{i=1}^{N}y_i^k\geq(\sum_{i=1}^{N}y_i)^k$.
2. if $k>1$, then $\sum_{i=1}^{N}y_i^k\geq N^{1-k}(\sum_{i=1}^{N}y_i)^k$.

Definition 2 [33]. Consider a system described as $\dot{x} = f(x), f(0) = 0$, with the origin as the equilibrium point, where $x \in \mathbb{R}^n$ and $f(x)$ is a nonlinear function. Provided that the origin is globally finite-time stable, it is fixed-time stable and the settling-time function $T(x_0)$ is limited, i.e., $\exists T_{max} > 0: T(x_0) \leq T_{max}, \forall\, x_0 \in \mathbb{R}^n$.

Definition 3 [34]. Convexity for a set $K \subset \mathbb{R}^{q\times 1}$ is achieved when $(1-\gamma)\upsilon + \gamma\omega \in K$ for $\upsilon, \omega \in K$ and $\gamma \in [0,1]$. Similarly, the convex hull spanned by the points $z_i \in \mathbb{R}^{q\times 1}, i = 1,2,\dots,m$ is defined as $Co\{z_1, z_2, \dots, z_m\} = \{\sum_{i=1}^{m} \alpha_i z_i \,|\alpha_i \in \mathbb{R}, \alpha_i \geq 0, \sum_{i=1}^{m} \alpha_i = 1\}$ where $Co(.)$ is the standard way of denoting the convex hull in literature (e.g. see [35]).

Lemma 2 [36]. Consider the differential equation $\dot{r} = -\alpha r^{\frac{m}{n}} - \beta r^{\frac{p}{q}}, \quad r(0) = r_0$, where α and β are positive real values, and m, n, p and q are positive odd integers. Also $n < m$ and $p < q$ are valid. The equilibrium point of this system is fixed-time stable and the settling time has the upper bound of $\frac{1}{\alpha\left(\frac{m}{n}-1\right)} + \frac{1}{\beta\left(1-\frac{p}{q}\right)}$.

To make it easier to use Lemma 2, m and n are taken as $2q - p$ and q, respectively. In this case, condition $n < m$ is also fulfilled.

3. Problem formulation

Assume that there is a group of N identical vehicles with M $(M < N)$ followers and $N - M$ leaders. Define the followers set by $\boldsymbol{\mathcal{F}} = \{1,2,\dots,M\}$ and leaders set by $\boldsymbol{\mathcal{L}} = \{M+1, M+2, \dots, N\}$. If an agent has no neighbors, it is called a leader. The fractional-order dynamics of the followers and the leaders will be explained as follows:

$$D^{\rho} x_i(t) = u_i(t), \quad i = 1,2,\dots,N, \tag{1}$$

where $0 < \rho < 1$ is the fractional order. If $\rho = 1$, the dynamics will reduce to integer order single integrators. Furthermore, $x_i \in \mathbb{R}^M$ is the state of the agent i and u_i is the control input of the follower $i \in \boldsymbol{\mathcal{F}}$ and the leader $i \in \boldsymbol{\mathcal{L}}$. We will consider the leaders with zero input and non-zero input. The Laplacian matrix L with $L_1 \in R^{M\times M}$ and $L_2 \in R^{M\times(N-M)}$ is defined as:

$$L = \begin{bmatrix} L_1 & L_2 \\ 0_{(N-M)\times M} & 0_{(N-M)\times(N-M)} \end{bmatrix}.$$

Assumption 1. The Laplacian matrix is assumed to be fixed. It is also assumed that the interaction graph consists of a spanning tree.

Assumption 2. There is one leader at least for each follower that has a directed path to that follower.

Lemma 3 [37]. Assumption 2 suggests that all the eigenvalues of L_1 are positive and each entry of $-{L_1}^{-1}L_2$ is nonnegative. The summation of the entries on each row of $-{L_1}^{-1}L_2$ is 1.

Assumption 3. The leader control inputs are limited by a constant value $\eta > 0$, satisfying $\|u_i(t)\| \leq \eta, \; i = M+1, M+2, \dots, N$.

Let $x_f = [x_1^T, x_2^T, \dots, x_M^T]^T$ and $x_l = [x_{M+1}^T, x_{M+2}^T, \dots, x_N^T]^T$. Determine the global containment error as:

$$\xi \coloneqq x_f + \left({L_1}^{-1}L_2 \otimes I_n\right) x_l, \tag{2}$$

where $\xi = [\xi_1^T, \xi_2^T, \dots, \xi_M^T]^T$ and $\otimes$ denotes the Kronecker product. We have:

$$\xi_i = x_i + \sum_{j=1}^{N-M} k_{ij} x_{M+j}, \tag{3}$$

for each $i \in \boldsymbol{\mathcal{F}}$ where k_{ij} is the (i,j)th element of ${L_1}^{-1}L_2$ satisfying:

$$-k_{ij} \geq 0, \; \sum_{j=1}^{N-M} -k_{ij} = 1. \tag{4}$$

Thus, if $\xi_i \to 0$ as $t \to \infty$, then we have $x_i \to \sum_{j=1}^{N-M} -k_{ij} x_{M+j}$. It also shows that follower i converges with coefficients $-k_{ij}, \; j = 1,2,\dots,N-M$ to the convex hull constructed by the leaders. Then, we consider ξ as the containment error. In other words, the control input for the followers is designed to force the states of the followers to go towards the convex hull, denoted by $\left(-({L_1}^{-1}L_2) \otimes I_n\right) x_l$. According to Definition 3, the states of the leaders construct this convex hull $(Co\{x_{M+1}^T, x_{M+2}^T, \dots, x_N^T\})$.

Remark 1. Applying the fractional-order Reimann-Liouville derivative of order ρ (given in Definition 1) to (3) we obtain:

$$D^{\rho}\xi_i = D^{\rho}\left(x_i + \sum_{j=1}^{N-M} k_{ij} x_{M+j}\right) = D^{\rho} x_i + \sum_{j=1}^{N-M} k_{ij} D^{\rho} x_{M+j} = u_i + \sum_{j=1}^{N-M} k_{ij} u_{M+j}. \tag{5}$$

In order to make (5) simpler, we define:

$$\omega_{ci} \coloneqq -\sum_{j=1}^{N-M} k_{ij} u_{M+j}, \; i = 1,2,\dots,M. \tag{6}$$

Then we have:

$$D^{\rho}\xi_i = u_i - \omega_{ci}. \tag{7}$$

The compact form of the containment error (7) is as follows:

$$D^{\rho}\xi = u - \omega_c. \tag{8}$$

here $u = [u_1^T, u_2^T, \dots, u_M^T]^T$ and $\omega_c = [\omega_{c1}^T, \omega_{c2}^T, \dots, \omega_{cM}^T]^T$.

4. Containment Tracking of Fractional-order MVSs

Consider the MVS given in (1). We propose the sliding function as:

$$s_i = D^{\rho-1}\xi_i + D^{\rho-2}\left[\alpha_1 sig(\xi_i)^{\frac{2q_1-p_1}{q_1}} + \beta_1 sig(\xi_i)^{\frac{p_1}{q_1}}\right], \quad i \in \{1,2,\dots,M\}, \tag{9}$$

where $sig(\cdot)^k = |\cdot|^k sign(\cdot)$ with $|\cdot|^k$ being the element-wise absolute value of vector to the power of k, and the parameters α_1 and β_1 are positive real numbers, and p_1 and q_1 are positive odd integers that satisfy $p_1 < q_1$. Thus $\frac{2q_1-p_1}{q_1} \geq 1$ and $\frac{p_1}{q_1} \leq 1$. Also, since $0 < \rho < 1$, then $\rho - 1$and $\rho - 2$ denote fractional integration. At the sliding surface:

$$s_i = 0 \;\to D^{\rho-1}\xi_i = -D^{\rho-2}\left[\alpha_1 sig(\xi_i)^{\frac{2q_1-p_1}{q_1}} + \beta_1 sig(\xi_i)^{\frac{p_1}{q_1}}\right], \tag{10}$$

Also, we have:

$$D^{\rho}\xi_i = -D^{\rho-1}\left[\alpha_1 sig(\xi_i)^{\frac{2q_1-p_1}{q_1}} + \beta_1 sig(\xi_i)^{\frac{p_1}{q_1}}\right] \to \dot{s} = 0, \tag{11}$$

Theorem 1. The containment tracking error system (11) is fixed-time stable and the upper limit is given by the following expression:

$$T_1 \leq \left(\frac{1}{\alpha_1 N^{\frac{p_1-q_1}{q_1}}} + \frac{1}{\beta_1}\right)\frac{q_1}{q_1-p_1}, \tag{12}$$

where $p_1 < q_1$.

Proof. Taking into account the Lyapunov function $V_1(t, x(t)) = \sum_{i=1}^{N} |\xi_i|$ with $x \in R^N$ and calculating the integer-order derivative of $V_1(t, x(t))$, we obtain:

$\dot{V}_1 = \sum_{i=1}^{N} sign(\xi_i)\dot{\xi}_i = \sum_{i=1}^{N} sign(\xi_i)D^{1-\rho}D^{\rho}\xi_i$

According to (11), it is achieved:

$$\dot{V}_1 = -\sum_{i=1}^{N} sign(\xi_i)D^{1-\rho}D^{\rho-1}\left[\alpha_1 sig(\xi_i)^{\frac{2q_1-p_1}{q_1}} + \beta_1 sig(\xi_i)^{\frac{p_1}{q_1}}\right] = -\sum_{i=1}^{N} sign(\xi_i)\left[\alpha_1 sig(\xi_i)^{\frac{2q_1-p_1}{q_1}} + \beta_1 sig(\xi_i)^{\frac{p_1}{q_1}}\right] = -\sum_{i=1}^{N} sign(\xi_i)\left[\alpha_1 sign(\xi_i)|\xi_i|^{\frac{2q_1-p_1}{q_1}} + \beta_1 sign(\xi_i)|\xi_i|^{\frac{p_1}{q_1}}\right] = -\sum_{i=1}^{N}\left[\alpha_1|\xi_i|^{\frac{2q_1-p_1}{q_1}} + \beta_1|\xi_i|^{\frac{p_1}{q_1}}\right] = -\sum_{i=1}^{N}\left[\alpha_1|\xi_i|^{\frac{2q_1-p_1}{q_1}} + \beta_1|\xi_i|^{\frac{p_1}{q_1}}\right] = -\alpha_1\sum_{i=1}^{N}|\xi_i|^{\frac{2q_1-p_1}{q_1}} - \beta_1\sum_{i=1}^{N}|\xi_i|^{\frac{p_1}{q_1}}$$

Considering Lemma **1**, the above equation yields:

$$\dot{V}_1 \le -\alpha_1 N^{1-\frac{2q_1-p_1}{q_1}}\left(\sum_{i=1}^{N}|\xi_i|\right)^{\frac{2q_1-p_1}{q_1}} - \beta_1\left(\sum_{i=1}^{N}|\xi_i|\right)^{\frac{p_1}{q_1}} = -\alpha_1 N^{\frac{p_1-q_1}{q_1}} V_1^{\frac{2q_1-p_1}{q_1}} - \beta_1 V_1^{\frac{p_1}{q_1}}$$

Using Lemma 2 in the above inequality with $\alpha = \alpha_1 N^{\frac{p_1-q_1}{q_1}}$, $\beta = \beta_1$, $p = p_1$, $q = q_1$, we obtain (12) which completes the proof. ■

The sliding mode containment control law is presented as follows:

$$u_i = \omega_{ci} - D^{\rho-1}\left(\alpha_1 sig(\xi_i)^{\frac{2q_1-p_1}{q_1}} + \beta_1 sig(\xi_i)^{\frac{p_1}{q_1}}\right) - \left(\alpha_2 sig(s_i)^{\frac{2q_2-p_2}{q_2}} + \beta_2 sig(s_i)^{\frac{p_2}{q_2}}\right), \tag{13}$$

where the parameters α_1, β_1, α_2 and β_2 are positive real numbers, and p_1, q_1, p_2 and q_2 are positive odd integers. Also $p_1 < q_1$ and $p_2 < q_2$.

The compact form (13) becomes:

$$u = \omega_c - D^{\rho-1}\left(\alpha_1 sig(\xi)^{\frac{2q_1-p_1}{q_1}} + \beta_1 sig(\xi)^{\frac{p_1}{q_1}}\right) - \left(\alpha_2 sig(S)^{\frac{2q_2-p_2}{q_2}} + \beta_2 sig(S)^{\frac{p_2}{q_2}}\right), \tag{14}$$

where $u = [u_1^T, u_2^T, \ldots, u_M^T]^T$ and $S = [s_1^T, s_2^T, \ldots, s_M^T]^T$.

Theorem 2. If for a MVS in the form of (1) the sliding function (9) holds, the control law (13) can solve the containment tracking problem for these systems with the settling time as follows:

$$T_2 \le \left(\frac{1}{\alpha_2} + \frac{1}{\beta_2}\right)\frac{q_2}{q_2-p_2}, \tag{15}$$

where $p_2 < q_2$.

Proof. Suppose the Lyapunov function as follows:

$$V_2(t, x(t)) = |S|, \tag{16}$$

where $x \in R^N$. Calculating the time derivate of $V_2(t, x(t))$, we obtain:

$$\dot{V}_2 = sign(S)\dot{S} = sign(S)\left(D^{\rho}\xi + D^{\rho-1}\left[\alpha_1 sig(\xi)^{\frac{2q_1-p_1}{q_1}} + \beta_1 sig(\xi)^{\frac{p_1}{q_1}}\right]\right). \tag{17}$$

Substituting (8) into (17), we obtain the following:

$$\dot{V}_2 = sign(S)\left(u - \omega_c + D^{\rho-1}\left[\alpha_1 sig(\xi)^{\frac{2q_1-p_1}{q_1}} + \beta_1 sig(\xi)^{\frac{p_1}{q_1}}\right]\right).$$

Substituting (14) in the above equation yields:

$$\dot{V}_2 = -sign(S)\left(\alpha_2 sig(S)^{\frac{2q_2-p_2}{q_2}} + \beta_2 sig(S)^{\frac{p_2}{q_2}}\right) = -sign(S)\left(\alpha_2 sign(S)|S|^{\frac{2q_2-p_2}{q_2}} + \beta_2 sign(S)|S|^{\frac{p_2}{q_2}}\right) = -\alpha_2|S|^{\frac{2q_2-p_2}{q_2}} - \beta_2|S|^{\frac{p_2}{q_2}} = -\alpha_2 V_2^{\frac{2q_2-p_2}{q_2}} - \beta_2 V_2^{\frac{p_2}{q_2}}.$$

Using Lemma 2 in the above inequality with $\alpha = \alpha_2$, $\beta = \beta_2$, $p = p_2$ and $q = q_2$, we obtain (15). ■

Remark 2. Consider fractional MVSs introduced in (1) with the sliding function (9). We can conclude from Theorem 1 and Theorem 2 that the containment control protocol (13), will cause the containment error to converge to zero within a fixed time, with the upper limit of:

$$T = T_1 + T_2 \le \left(\frac{1}{\alpha_1 N^{\frac{p_1-q_1}{q_1}}} + \frac{1}{\beta_1}\right)\frac{q_1}{q_1-p_1} + \left(\frac{1}{\alpha_2} + \frac{1}{\beta_2}\right)\frac{q_2}{q_2-p_2}. \tag{18}$$

Remark 3. The preceding equations can be extended to the case that the states of the vehicles are vectors rather than scalars via Kronecker product.

Remark 4. From (18), it is obvious that $p_1 < q_1$ and $p_2 < q_2$ must be satisfied to achieve fixed-time containment. The other parameters, i.e. α_1, α_2, β_1 and β_2 along with p_1, p_2, q_1 and q_2 can be selected with the designer's choice to obtain the desired upper bound for the settling time T in (18). Additionally, the mentioned parameters also appear in the control signal (13) which means that any change in the settling time necessitates making changes to the control signal. It is clear that a trade-off between the settling time and smoothness of the control signal might be necessary to obtain fixed-time stability along with acceptable performance.

5. Formation-Containment of Fractional-order MVSs

In order to obtain a solution for the formation-containment problem, a continuous function is defined as $h(x_i) = x_i - h_i^F$ where the desired movement of vehicle i is h_i^F. For $h_i^F = 0$, the protocol will be a containment algorithm. If h_i^F is time-invariant, the sliding function (9) and the protocol given in (13) will bring about formation-containment.

The followers are described with (1). If the containment error for containment is (3), then we can define it for formation-containment as:

$$\xi_i = h(x_i) + \sum_{j=1}^{N-M} k_{ij}\left(h(x_{M+j})\right) = x_i - h_i^F + \sum_{j=1}^{N-M} k_{ij}\left(x_{M+j} - h_{M+j}^F\right). \tag{19}$$

Also, the derivation of (19) is:

$$D^{\rho}\xi_i = D^{\rho}\left(x_i - h_i^F + \sum_{j=1}^{N-M} k_{ij}\left(x_{M+j} - h_{M+j}^F\right)\right) = D^{\rho}x_i - D^{\rho}h_i^F + \sum_{j=1}^{N-M} k_{ij}D^{\rho}x_{M+j} - \sum_{j=1}^{N-M} k_{ij}D^{\rho}h_{M+j}^F = u_i - D^{\rho}h_i^F + \sum_{j=1}^{N-M} k_{ij}u_{M+j} - \sum_{j=1}^{N-M} k_{ij}D^{\rho}h_{M+j}^F. \tag{20}$$

In order to simplify (20), we can define:

$$\omega_{hi} := -\sum_{j=1}^{N-M} k_{ij}D^{\rho}h_{M+j}^F, \quad i = 1,2,\ldots,M. \tag{21}$$

According to (6), (21) and (20) we obtain:

$$D^{\rho}\xi_i = u_i - D^{\rho}h_i^F - \omega_{ci} + \omega_{hi}, \tag{22}$$

and the compact form of the (22) is as follows:

$$D^{\rho}\xi = u - D^{\rho}h^F - \omega_c + \omega_h, \tag{23}$$

where $h^F = \left[{h_1^F}^T, {h_2^F}^T, \ldots, {h_M^F}^T\right]^T$. For the formation-containment problem, the protocol will be:

$$u_i = D^{\rho}h_i^F + \omega_{ci} - \omega_{hi} - D^{\rho-1}\left(\alpha_1 sig(\xi_i)^{\frac{2q_1-p_1}{q_1}} + \beta_1 sig(\xi_i)^{\frac{p_1}{q_1}}\right) - \tag{24}$$

$\left(\alpha_2 sig(s_i)^{\frac{2q_2-p_2}{q_2}} + \beta_2 sig(s_i)^{\frac{p_2}{q_2}}\right)$.

The compact form of (24) becomes:

$$u = D^\rho h^F + \omega_c - \omega_h - D^{\rho-1}\left(\alpha_1 sig(\xi)^{\frac{2q_1-p_1}{q_1}} + \beta_1 sig(\xi)^{\frac{p_1}{q_1}}\right) - \left(\alpha_2 sig(S)^{\frac{2q_2-p_2}{q_2}} + \beta_2 sig(S)^{\frac{p_2}{q_2}}\right). \quad (25)$$

Theorem 3. For the MVS (1) with the proposed protocol (24) and a formation-containment vector $h(x_i)$, the formation-containment with the settling time upper limited by (15) is achievable.

Proof. Considering the Lyapunov function (16) and substituting (23) into (17), we obtain the following:

$$\dot{V}_2 = sign(S)\left(u - D^\rho h^F - \omega_c + \omega_h + D^{\rho-1}\left[\alpha_1 sig(\xi)^{\frac{2q_1-p_1}{q_1}} + \beta_1 sig(\xi)^{\frac{p_1}{q_1}}\right]\right).$$

Substituting (25) in the above equation yields:

$$\dot{V}_2 = -sign(S)\left(\alpha_2 sig(S)^{\frac{2q_2-p_2}{q_2}} + \beta_2 sig(S)^{\frac{p_2}{q_2}}\right) = -sign(S)\left(\alpha_2 sign(S)|S|^{\frac{2q_2-p_2}{q_2}} + \beta_2 sign(S)|S|^{\frac{p_2}{q_2}}\right) = -\alpha_2|S|^{\frac{2q_2-p_2}{q_2}} - \beta_2|S|^{\frac{p_2}{q_2}} = -\alpha_2 V_2^{\frac{2q_2-p_2}{q_2}} - \beta_2 V_2^{\frac{p_2}{q_2}}.$$

Using Lemma 2 in the above inequality with $\alpha = \alpha_2$, $\beta = \beta_2$, $p = p_2$ and $q = q_2$, the (15) is obtained and the proof is complete. ■

Remark 5. Assume a MVS in (1) and the sliding function (9). It is resulted from Theorem 1 and Theorem 3 that the vehicles converge to a certain formation-containment with the control law of (24) and the error system will be fixed-time stable.

Remark 6. The proposed containment protocol possesses the properties of anti-disturbance and robustness to uncertainties which have been inherited from SMC approach. Note that the sliding surface presented in (9) is a class of integral sliding mode surface where adding the fractional integration of errors into the sliding surface leads to a much better anti-disturbance characteristics. In (9), $D^{\rho-2}$ denotes fractional integration with the order of $2-\rho$ which reduces to classical integrator for $\rho = 1$.

Remark 7. The use of a sign function can contribute to chattering in control systems. Numerous methods have been suggested to cope with this issue such as introducing a dead-band or hysteresis around the control value. Although sign function has been employed in the control protocol (13), since the fractional integrator $D^{\rho-1}$ appears in the control signal, it acts as a low-pass filter which results in a quite smooth control signal.

Remark 8. When robots cooperate with each other in complex environments such as viscous fluids, the integer order dynamics such as first order dynamics cannot describe the motion of the robot accurately. For example, consider first order integer dynamics for robots as $D^\rho x_i(t) = u_i(t), \quad i = 1,2,\dots,N$ where $\rho = 1$ and N is the number of agents. If $x_i(t)$ is the position of the robot along a specific axis, then $u_i(t)$ is the velocity of the robot along the same axis. This is true in normal environments. However, when the environment is viscous, the velocity of the robot does not equal the first-order derivative of the position. In fact, the velocity will be the fractional derivative of the position which is denoted by $D^\rho x_i(t) = u_i(t), \ i = 1,2,\dots,N$ where ρ is not an integer anymore and it is defined according to the viscosity of the fluid. When multiple robots cooperate in such environments, the dynamics of the agents are best modeled by fractional-order dynamics. In the movement of the robots in viscous areas, the leaders which are mostly equipped with high-quality sensors make a formation of convex hull, inside which the followers are placed. As the formation of the leaders move, the followers which are inside the convex hull as the safe area move along with the leaders provided that the containment protocol works well.

6. Formation-Containment of Fractional-order MVSs with External Disturbance

Supposed that the fractional-order dynamics of the followers with external disturbances is as follows:

$$D^\rho x_i(t) = u_i(t) + \delta_i(t), \quad i \in \{1,2,\dots,M\}, \quad (26)$$

where $\delta_i \in R^n$ refers to the external disturbance of vehicle i. The leaders are supposed to be in the form of (1).

Assumption 4. We assume that the external disturbance denoted by $\delta_i(t)$ satisfies $|\delta_i(t)| \le \gamma_i$. As a result, its compact form can be written as $|\delta(t)| \le \gamma$.

Thus for the disturbed system given in (26), the fractional derivative of (19) is:

$$D^\rho \xi_i = D^\rho\left(x_i - h_i^F + \sum_{j=1}^{N-M} k_{ij}\left(x_{M+j} - h_{M+j}^F\right)\right) = D^\rho x_i - D^\rho h_i^F + \sum_{j=1}^{N-M} k_{ij} D^\rho x_{M+j} - \sum_{j=1}^{N-M} k_{ij} D^\rho h_{M+j}^F = u_i + \delta_i - D^\rho h_i^F + \sum_{j=1}^{N-M} k_{ij}\left(u_{M+j} + \delta_{M+j}\right) - \sum_{j=1}^{N-M} k_{ij} D^\rho h_{M+j}^F = u_i + \delta_i - D^\rho h_i^F + \sum_{j=1}^{N-M} k_{ij} u_{M+j} + \sum_{j=1}^{N-M} k_{ij}\delta_{M+j} - \sum_{j=1}^{N-M} k_{ij} D^\rho h_{M+j}^F. \quad (27)$$

In order to simplify (27), we can define:

$$\omega_{\delta i} := -\sum_{j=1}^{N-M} k_{ij}\delta_{M+j}, \ i = 1,2,\dots,M. \quad (28)$$

According to (6), (21), (28) and (27), we have:

$$D^\rho \xi_i = u_i + \delta_i - D^\rho h_i^F - \omega_{ci} - \omega_{\delta i} + \omega_{hi}, \quad (29)$$

and the compact form of the (29) is as follows:

$$D^\rho \xi = u + \delta - D^\rho h^F - \omega_c - \omega_\delta + \omega_h. \quad (30)$$

The formation-containment control law by considering the external disturbances, is proposed as:

$$u_i = D^\rho h_i^F + \omega_{ci} - \omega_{hi} - D^{\rho-1}\left(\alpha_1 sig(\xi_i)^{\frac{2q_1-p_1}{q_1}} + \beta_1 sig(\xi_i)^{\frac{p_1}{q_1}}\right) - \left(\alpha_2 sig(s_i)^{\frac{2q_2-p_2}{q_2}} + \beta_2 sig(s_i)^{\frac{p_2}{q_2}}\right) - \gamma_i sign(s_i), \quad (31)$$

where the parameters α_1, β_1, α_2 and β_2 are positive real values, and p_1, q_1, p_2 and q_2 are positive odd integers satisfying $p_1 < q_1$ and $p_2 < q_2$.

The compact form of (31) becomes:

$$u = D^\rho h^F + \omega_c - \omega_h - D^{\rho-1}\left(\alpha_1 sig(\xi)^{\frac{2q_1-p_1}{q_1}} + \beta_1 sig(\xi)^{\frac{p_1}{q_1}}\right) - \left(\alpha_2 sig(S)^{\frac{2q_2-p_2}{q_2}} + \beta_2 sig(S)^{\frac{p_2}{q_2}}\right) - \gamma sign(S). \quad (32)$$

Theorem 4. The control protocol (31) guarantees achieving formation-containment with the sliding function (9) for the vehicles with the dynamics of fractional order with disturbance given in (26). The upper bound of the settling time has been proposed in (15).

Proof. With the Lyapunov function (16) and substituting (30) into (17), we obtain the following:

$$\dot{V}_2 = sign(S)\Big(u + \delta - D^{\rho}h^{F} - \omega_c - \omega_\delta + \omega_h + D^{\rho-1}\Big[\alpha_1 sig(\xi)^{\frac{2q_1-p_1}{q_1}} + \beta_1 sig(\xi)^{\frac{p_1}{q_1}}\Big]\Big).$$

Substituting (32) in the above equation yields:

$$\dot{V}_2 = sign(S)\Big(\delta - \omega_\delta - \gamma sign(S) - \Big(\alpha_2 sig(S)^{\frac{2q_2-p_2}{q_2}} + \beta_2 sig(S)^{\frac{p_2}{q_2}}\Big)\Big) = sign(S)\big(\delta - \omega_\delta - \gamma sign(S)\big) - sign(S)\Big(\alpha_2 sig(S)^{\frac{2q_2-p_2}{q_2}} + \beta_2 sig(S)^{\frac{p_2}{q_2}}\Big) = -(\gamma - sign(S)\delta + sign(S)\omega_\delta) - sign(S)\Big(\alpha_2 sign(S)|S|^{\frac{2q_2-p_2}{q_2}} + \beta_2 sign(S)|S|^{\frac{p_2}{q_2}}\Big) \le -(\gamma - |\delta| + |\omega_\delta|) - \alpha_2|S|^{\frac{2q_2-p_2}{q_2}} - \beta_2|S|^{\frac{p_2}{q_2}}.$$

According to Assumption 4 and $|\omega_\delta| \ge 0$, then $\gamma - |\delta| + |\omega_\delta| \ge 0$ and we obtain:

$$\dot{V}_2 \le -\alpha_2|S|^{\frac{2q_2-p_2}{q_2}} - \beta_2|S|^{\frac{p_2}{q_2}} = -\alpha_2 V_2^{\frac{2q_2-p_2}{q_2}} - \beta_2 V_2^{\frac{p_2}{q_2}}.$$

Based on Lemma 2, it is achieved that:

$$\begin{cases} \alpha = \alpha_2 \quad and \quad \beta = \beta_2 \\ p = p_2 \quad and \quad q = q_2 \end{cases}.$$

Then we obtain (15) which completes the proof. ■

7. Simulations Results with fixed Leaders

When robots cooperate with each other in complex environments such as viscous fluids, dynamics of integer orders cannot describe the motion of the robot accurately. For example, consider first order integer dynamics for robots as $D^{\rho}x_i(t) = u_i(t), \quad i = 1,2,\dots,N$ where $\rho = 1$ and N is the number of vehicles. If $x_i(t)$ is the position of the robot along a specific axis, then $u_i(t)$ is the velocity of the robot along the same axis. This is true in normal environments. However, when the environment is viscous, the velocity of the robot does not equal the first-order derivative of the position. In fact, the velocity will be the fractional derivative of the position which is denoted by $D^{\rho}x_i(t) = u_i(t), \quad i = 1,2,\dots,N$ (as defined in (1)) where ρ is not an integer anymore and it is defined according to the viscosity of the fluid. When multiple robots cooperate in such environments, the dynamics of the vehicles are best modeled by fractional-order dynamics. Some numerical simulations are supplied to confirm the theoretical results in this section. Assume a fractional-order MVS consisting of seven vehicles described by (1) as explained above. The vehicle interaction topology is displayed in Fig. 1 where the labels of followers are {1,2,3,4,5} and the labels of the leaders are {6,7}. The Laplacian matrix is as follows:

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where L_1 and L_2 are as:

$$L_1 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

The preliminary conditions of the followers are selected as $x_0 = [-3,2,-1.5,4.5,-1,0.5,-0.5]$. The parameters of (13) are $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 2$, $p = 7$, and $q = 9$ for each protocol and the fractional order is selected as $\rho = 0.9$ for all of the simulations. Also, the upper limit of the settling time according to (18), is calculated as $T_{fixed-time} = 9.96$.

First, we investigate the containment tracking of the vehicles with the proposed approach. Achieving containment for each state variable is displayed in Fig. 2. In this figure the first states of the vehicles (positions along *x* axis) have been depicted. Fig. 3 illustrates that the containment error goes to zero in a fixed-time. The sliding function is also displayed in Fig. 4. According to these figures, the containment tracking has been achieved in a time which is upper limited by the obtained settling time $T_{fixed-time}$ and it is independent of the initial states.

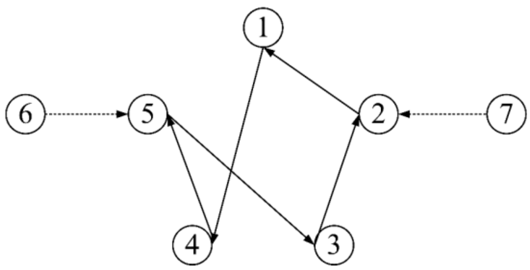

Fig. 1. The topology of the vehicle interaction.

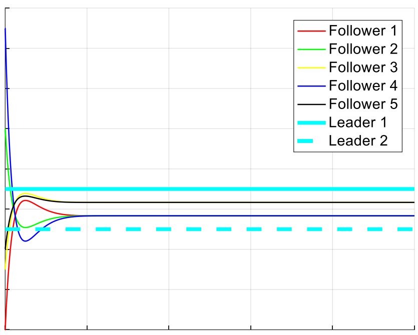

Fig. 2. Containment of the vehicles.

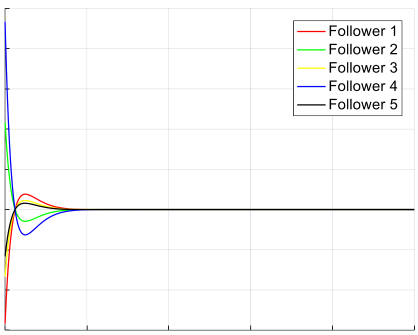

Fig. 3. Containment errors.

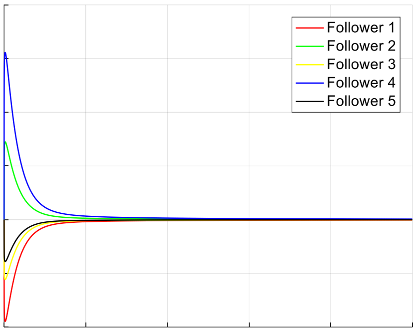

Fig. 4. Sliding function.

Next, we examine the performance of the formation-containment algorithm. The goal is achieving time-invariant and time-variant sinusoidal formations given as $h_i^F = 0.2\,cos(\pi + \frac{k_i\pi}{7})$, $k_i = \{1,2,\dots,7\}$ and $h_i^F = 0.2\,cos(\pi t + \frac{k_i\pi}{7})$, $k_i = \{1,2,\dots,7\}$ for all seven vehicles. Fig. 5 and Fig. 6 show how the states of the vehicles converge with the prescribed formations. The upper limit of the settling time remains the same as before ($T_{fixed-time}$). It is concluded from these figures that the vehicles can perfectly maintain their formation-containment in the calculated fixed-time as well as converge to the convex hull formed by the leaders' states.

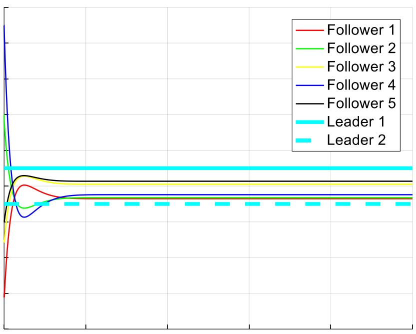

Fig. 5. Formation-containment with $\boldsymbol{h_i^F = 0.2\,cos(\pi + \frac{k_i\pi}{7})}$.

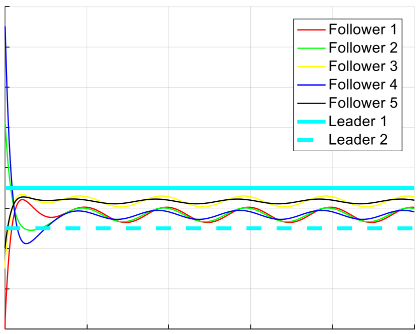

Fig. 6. Formation-containment with $\boldsymbol{h_i^F = 0.2\,cos(\pi t + \frac{k_i\pi}{7})}$.

Then, we investigate the performance of the containment algorithm in presence of external disturbance. The parameters are considered the same as before and the disturbances applied to the vehicles at the instant of 5 seconds are:

$w_1 = 0.1\,cos(7t)$, $w_2 = 0.3\,cos(6t + \frac{\pi}{2})$, $w_3 = 0.1\,cos(5t)$, $w_4 = 0.3\,cos(4t - \frac{\pi}{2})$, $w_5 = 0.1\,cos(3t)$, $w_6 = 0.3\,cos(2t + \frac{\pi}{2})$, $w_7 = 0.1\,cos(t)$.

In addition, with the given disturbances, the parameter γ is equal to 1. The containment is achieved within the same settling time upper bounded by $T_{fixed-time}$. Fig. 7 and Fig. 8 illustrate the containment error and the formation-containment error for the disturbed MVS, respectively. The formation-containment is selected time-variant similar to the previous simulation. We can conclude according to these figures that the vehicles have perfectly achieved containment in presence of disturbances.

8. Simulations Results with Dynamic Leaders

All simulations are the same as the previous section with minor differences. The dynamics of the leaders are selected as $u_6 = cos(t)$ and $u_7 = sin(2t)$. The containment of the vehicles is displayed in Fig. 9. Also, Fig. 10 shows the containment errors which converge to zero in a fixed time. In order to illustrate that the proposed method also works in larger networks, 30 followers with two dynamic leaders with $u = cos(5t)$ and $u = sin(t)$ have been tested and the results are depicted in Fig. 11. As it is shown, the states of the 30 followers stay in the convex hull made by the leaders' states and the containment has been achieved. In addition, the simulation of applying the proposed protocol to the formation-containment problem is displayed in Fig. 12 and Fig. 13 with two different formations.

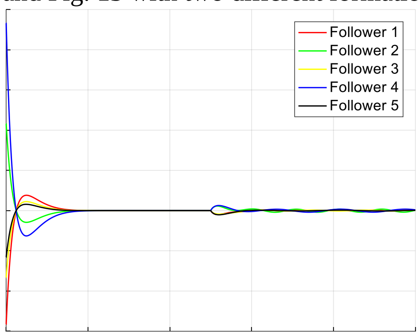

Fig. 7. Containment error of vehicles with external disturbance.

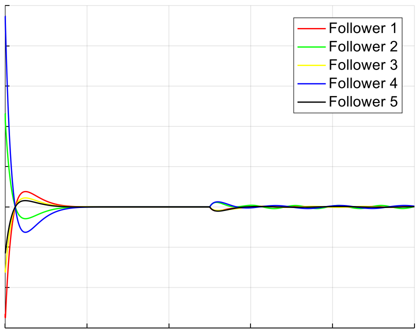

Fig. 8. Formation-containment error of vehicles with external disturbance.

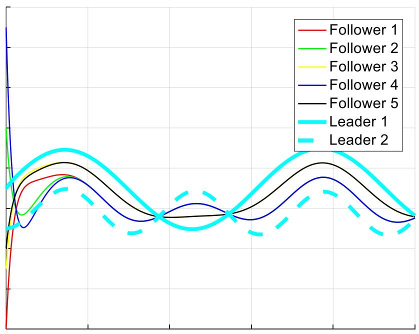

Fig. 9. Containment with dynamic leaders.

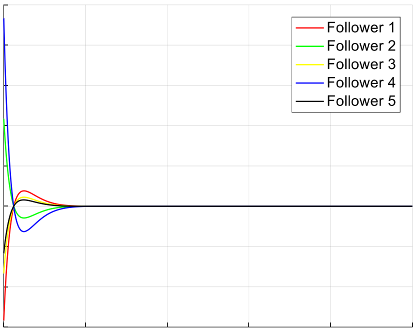

Fig. 10. Containment error with dynamic leaders.

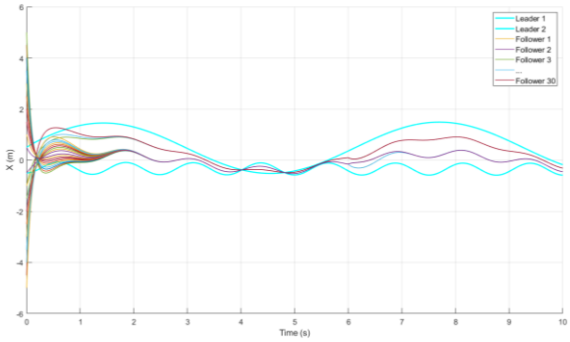

Fig. 11. Containment in a large network with 30 followers and two dynamic leaders.

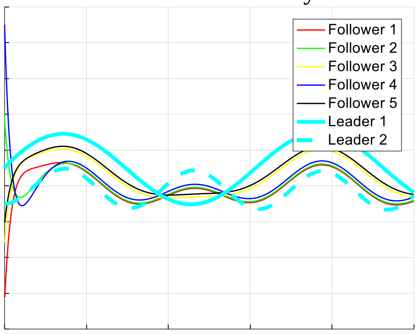

Fig. 12. Formation-containment with dynamic leaders and $h_i^F = 0.2\,cos(\pi + \frac{k_i\pi}{7})$.

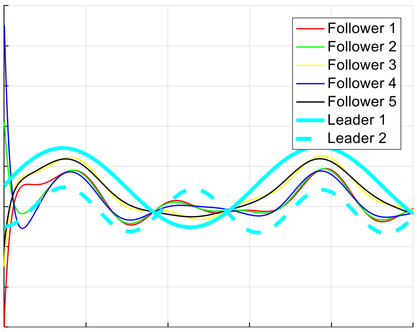

Fig. 13. Formation-containment with dynamic leaders and $h_i^F = 0.2\,cos(\pi t + \frac{k_i\pi}{7})$.

Finally, to illustrate the convergence to the convex hull, we consider a different interaction topology with four followers and three leaders as shown in Fig. 14. The followers have been labeled by {1,2,3,4} and the label of leaders are {5,6,7}. The Laplacian matrix is as follows:

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and the two-dimensional preliminary conditions of the followers are selected as:

$$x_0 = [(-3,1.5),(2,2.5),(-1.5,-4),(4.5,-5),\\ (-1,-0.5),(0.5,1),(-0.5,0.5)]\,,$$

and the dynamics of the leaders are selected as $u_5 = 2sin(t)$, $u_6 = cos(t)$ and $u_7 = sin(2t)$. Fig. 15 shows the containment tracking results with dynamic leaders where the positions of the leaders and the followers along *x* and *y* axes are shown on the *x*-*y* plane at different times. It is illustrated that the containment tracking is fully achieved in 1.5 seconds. These figures were depicted in 1.5 seconds, every 0.3 of which is displayed until the agents have converged. It is seen that a moving convex hull is successfully made by the 3 leaders, and the 4 followers are contained in the convex hull.

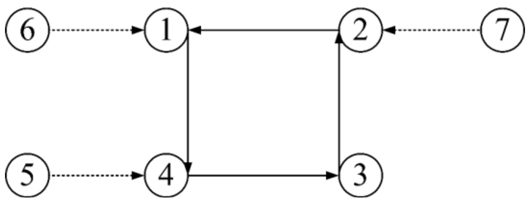

Fig. 14. The new interaction topology of the vehicles.

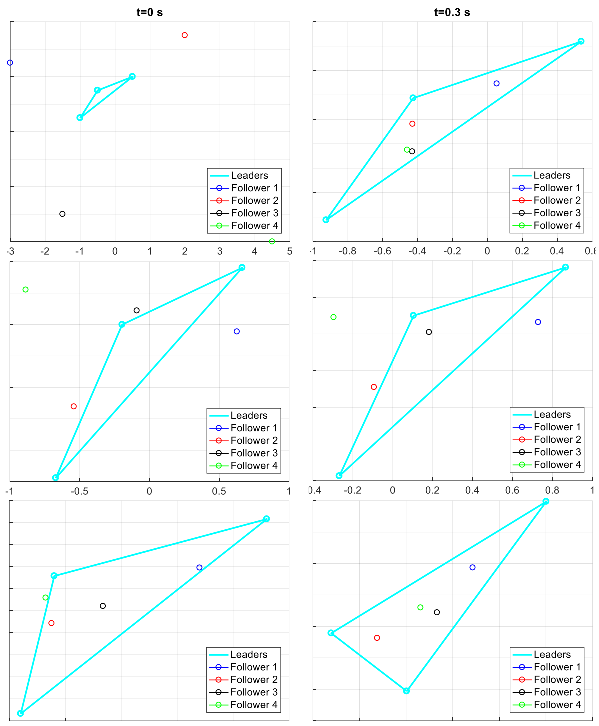

Fig. 15. Containment tracking with moving convex hull.

9. Conclusions

This article addresses achieving containment for fractional order MVSs with multiple leaders within a certain settling time. We proposed a novel SMC protocol where the states of the followers move to the convex hull spanned by the leaders' states. The novelty of the SMC approach proposed in this paper lies in the fact that the convergence time of the containment achievement is not affected by the initial conditions of the vehicles' states. Since the upper bound of the settling time can be tubed by the designer, a remarkable merit has been achieved from practical point of view. Additionally, the external disturbances were very well dealt with. The results in this paper have been presented for fractional-order MVSs which means that the results can also be applied to integer-order MVSs by assuming the fractional order to be one. The simulations results illustrated that the containment tracking and also containment-formation with dynamic leaders were successfully achieved where the upper limits of the settling times were obtained beforehand by the proposed relations.

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