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Robust Stability Analysis of Switching Time-Delay Systems via a Novel Non-Monotonic Lyapunov–Krasovskii Approach

Saeed Ahmadi[1], Younes Solgi[1,*]

[1] *Department of Electrical Engineering, Bu-Ali Sina University, Hamedan, Iran*

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ABSTRACT

In this paper, a novel approach based on a non-monotonic Lyapunov–Krasovskii functional is first proposed for the comprehensive stability analysis of discrete-time switched systems with time delays in an m-step-ahead scheme. Subsequently, by extending these results, non-uniform robust stability conditions suitable for systems with uncertainties are derived. In this method, the stringent requirement of the uniform decrease of the Lyapunov–Krasovskii functional is replaced by non-monotonic conditions; that is, the functional is allowed to increase at certain steps, while its overall trend must remain decreasing. Consequently, this approach accommodates a broader class of functionals for stability analysis. Within the non-monotonic Lyapunov–Krasovskii framework, a set of sufficient conditions in the form of linear matrix inequalities (LMIs) is formulated to assess global asymptotic stability of discrete-time delayed systems. Moreover, Abel's lemma is employed to further reduce conservatism in the stability analysis of switched systems compared to previous studies. Unlike its continuous-time counterpart, the discrete-time model exhibits higher complexity due to the interactions among subsystems induced by switching. To demonstrate the effectiveness of the proposed method, simulation results are provided for two numerical examples.

1. Introduction

Time delays are commonly encountered in numerous practical systems, including biological networks, neural systems, mechanical structures, and control applications [1]. Among the widely adopted strategies for examining the stability of time-delay systems is the Lyapunov-based framework, which has been extensively utilized in control engineering [2]–[12]. Lyapunov functions serve as classical instruments for analyzing the stability of dynamical systems, with bounding inequalities playing a crucial role in optimization and stability assessment [3]. Despite their broad use, selecting an appropriate monotonic Lyapunov function (MLF) that reduces conservatism for various system classes remains a challenging issue, particularly for systems with delays [4]. Both continuous- and discrete-time stability analyses frequently employ Lyapunov–Razumikhin and Lyapunov–Krasovskii functionals [5], [6]. Among these, the Lyapunov–Krasovskii functional (LKF) approach is preferred in many studies because it typically produces less conservative results than the Razumikhin method [7]. This method has been applied widely, including in model predictive control and trajectory-tracking problems [9]. A central aspect of the LKF method is the careful construction of a functional. Its main advantage is that sufficient stability conditions for time-delay systems can be expressed as linear matrix inequalities (LMIs), which allow precise mathematical treatment and efficient numerical optimization while avoiding unnecessary conservatism [10]. Significant research has focused on refining the structure of LKFs. The core idea is to construct a suitable functional and then derive conditions

* Corresponding author
E-mail address: y.solgi@basu.ac.ir
https://orcid.org/0000-0003-1279-0153
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ensuring the negativity condition of its forward difference. Over the past two decades, numerous LKF formulations have been developed and refined to achieve tighter estimates. These include summation inequalities [11], [12], free-weighting matrices [8], reciprocally convex combinations [11], and Jensen-based inequalitie (JBI) [13]–[38]. Moreover, Zhang et al. employed Abel's lemma to reduce conservatism in stability conditions by providing more accurate estimates for the LKF forward difference, especially in summation terms [14]. Collectively, these studies aim to improve the LKF framework to yield less conservative stability results. An alternative strategy to decrease conservatism in Lyapunov-based analysis is the non-monotonic Lyapunov method, also known as the finite-step Lyapunov approach [15], [16]. Unlike traditional Lyapunov techniques that require continuous decrease, this method allows the Lyapunov function to increase temporarily, as long as it decreases after a finite number of steps. This relaxation makes it possible to employ a broader spectrum of functions that can be used for assessing system stability. However, studies on discrete-time systems using this method remain limited. For example, in their work, Aylas and Peuteman modified some conditions to ensure global asymptotic convergence, thereby introducing the finite-step Lyapunov method. A related idea, known as non-quadratic stabilization, has been applied for stabilizing Takagi–Sugeno fuzzy systems [17]. Ahmadi and Parrilo proposed alternative criteria to ensure global asymptotic stability in discrete-time systems, naming their framework non-monotonic Lyapunov stability criterion [15], in agreement with [16]. Later, Derakhshani and Fathi introduced a discrete-time variant of the non-monotonic Lyapunov method for fuzzy control systems [18], [19]. Their approach reduces conservatism by easing the strict monotonicity constraint imposed in classical Lyapunov theory [20]. Building on this, Shi et al. extended the non-monotonic Lyapunov functional technique to discrete-time switched linear systems, which resulted in more flexible stability conditions and enhanced H∞ performance [21]. Another practical challenge is system uncertainty, which may cause variations in the model and lead to instability. Several studies have addressed discrete-time uncertain time-delay systems. For instance, [22] studied robust stability and stabilization of discrete-time systems subject to delays and bounded parametric uncertainties. In [23], robust stability was investigated for systems with state delays within the state-space framework using generalized overflow calculus. Similarly, [24] considered delay-dependent robust stability for uncertain discrete-time systems with time-varying delays and nonlinear disturbances, using the LKF approach.

In this paper, we propose a novel (NMLK) method for analyzing the stability of discrete-time switched linear systems with delays. The approach can handle both known and unknown constant delays, provides explicit upper bounds on admissible delays, and avoids the rapid increase in the number of Lyapunov functionals typically required in conventional stability proofs. Notably, this study demonstrates that, unlike previous works where the strict monotonic decrease condition often necessitates the use of six Lyapunov functionals or the repeated addition of extra terms to these functionals to ensure the stability of systems with complex trajectories, the proposed method effectively guarantees stability using only three Lyapunov functionals, while exhibiting significantly reduced conservatism compared to traditional approaches. This work extends the use of non-monotonic Lyapunov–Krasovskii functionals to the stability analysis of switched systems with delays in the dynamic structure. Another key contribution is the use of Abel's lemma for switched systems to further reduce conservatism. In the state-space formulation of switched time-delay systems, the delayed state $x(k-d)$ explicitly appears, complicating the treatment of intermediate states between $x(k)$ and $x(k-d)$. Additionally, summation terms involving past states introduce further complexity. Our proposed approach addresses these issues by extending the analysis to m-step delays, enabling effective stabilization of systems with input/output delays. Moreover, the robust stability of the proposed system is investigated through NMLK-derived conditions. Since uncertainties may introduce variations in the model, the Lyapunov–Krasovskii functional may temporarily increase, potentially threatening stability. The robust NMLK formulation overcomes this by not enforcing strict decrease at each step. New lemmas are introduced and integrated into a novel LMI derivation and the complete set of stabilization conditions and computational steps is provided.

The structure of this paper is as follows:

Section 2 presents the preliminary concepts related to this study. Section 3 includes the principal findings, which cover the m-step non-monotonic Lyapunov–Krasovskii-based stability conditions as well as the robust stability analysis in the presence of parametric uncertainties. Section 4 evaluates the proposed method through numerical examples and also presents the simulation results. Section 5 is devoted to the summary and conclusion.

Nomenclature

$\lambda_{max}(A)$: the largest eigenvalue associated with matrix A;

$\|\cdot\|$: euclidean vector norm;

$*$: symmetric terms in a symmetric matrix;

R (R^+): subsets of $\mathbb{R}$ consisting of strictly positive numbers;

$R^{n\times n}$:n-dimensional euclidean space;

$\|A\|_2 = \sqrt{\lambda_{max}A^TA}$ spectral norm;

2. Definitions and preliminaries

a. Non-Monotonic Lyapunov Stability Method

The discrete-time switched system under consideration is described as follows:

$$f: R^n \to R^n$$

$$x_{(k+1)} = f_{\sigma(k)}\left(x_{(k)}\right) \quad (1)$$

where $\sigma(k) = i$, $x(k) \in R^n$, $f: R^n \to R^n$ with $f(0) = 0$ To analyze the stability of the equilibrium point, the origin (zero) is considered as the coordinate origin [20].

The Lyapunov stability condition states that there exists a continuous scalar function$V(x_{(k)})$ such that the following conditions hold:

1)V($x_{(k)}$) >0 is a positive definite function (PDF);

2)V($x_{(k)}$) → ∞ as $x \to \infty$;

3)V(0) =0 The value of the Lyapunov function at the equilibrium point, which is taken as the origin, should be zero;

4)V$(x_{(k+1)})$ −V$(x_{(k)})$ < 0 for $x_{(k)} \neq 0$;

If all the conditions are satisfied, $V(x_{(k)})$ is a Lyapunov function [25].

Definition 1: The function $f(x)$ is globally Lipschitz if there exists a constant $L > 0$ such that:

$$\forall x, y \epsilon R^n, |f(x) - f(y)| \leq L|x - y| \tag{2}$$

This condition is introduced to indicate that the function, defined as the system dynamics, is bounded and well-behaved. The constant L is referred to as the Lipschitz constant. In stability proofs, the Lipschitz property helps ensure that the Lyapunov functions behave properly and are differentiable as required.

Theorem 1: *(Non-monotonic Lyapunov Stability)* [25]-[37]:

Consider the discrete system (1), where $x_{(k)} \in R^n$ and $f: R^n \to R^n$ is globally Lipschitz continuous with constant L satisfying $f(0) = 0$. If there exists a continuous scalar function V($x_{(k)}$) that satisfies the following conditions:

1) V($x_{(k)}$) >0 is a positive definite function (PDF);

2) V($x_{(k)}$) → ∞ as $x \to \infty$;

3) V(0)=0 The value of the Lyapunov function at the equilibrium point, which is taken as the origin, should be zero;

4) V$(x_{(k+m)})$ −V$(x_{(k)})$ < 0 for x(k) ≠ 0;

where m specifies the number of steps over which the Lyapunov function is allowed to increase. In this case, the system is stable at the origin.

b. Lyapunov-Krasovskii Stability Method

In time-delay systems, the Lyapunov function is influenced by both the present state and historical states. Consequently, the Lyapunov–Krasovskii functional is employed in place of the standard Lyapunov function for stability analysis The space of continuous functions C over the interval$[-\mathrm{d}, 0]$ is defined with the supremum norm |·|. Additionally, we define the following variable:

$$\bar{x} = (x_{(k)}^T, x_{(k-d)}^T)^T$$

The subsequent theorem introduces the Lyapunov–Krasovskii approach for stability analysis.

Theorem 2: *(Lyapunov-Krasovskii Stability)* [25]:

If there exists a continuous function $V: R^n \to R$ and two continuous, non-decreasing functions v and $w: R^n \to R^+$, such that:

$$v(0) = w(0) = 0, v(s) > 0, w(s) > 0 \; \forall \; s > 0$$

Now, if the following conditions hold for $x(k)$:

$$0 < V(x_{(k)}) \leq v\|\bar{x}\|^2, V(0) = 0 \tag{3}$$

$$\Delta V(x_{(k)}) = V(x_{(k+1)}) - V(x_{(k)}) \leq -w\|\bar{x}\|^2 \tag{4}$$

Then, the system's equilibrium point $x = 0$ is asymptotically stable.

c. Finite Sum Inequality Based on Abel's Lemma: Finally, the following lemma will be used in this paper to derive the NMLK-based stability criterion in order to reduce conservatism:

Lemma 1*. (Finite Sum Inequality Based on Abel's Lemma)* [20]:

Let $H \epsilon R^{n \times n}$ be a fixed positive definite matrix such that $H = H^T > 0$ and two integers r_1 and $r_2$such that $r_2 - r_1 > 1$, the following inequality holds:

$$\sum_{j=r_1}^{r_2-1} \eta_{(j)}^T H \eta_{(j)} \geq \frac{1}{\ell_1} v_1^T H v_1 + \frac{3\ell_2}{\ell_1 \ell_3} v_2^T H v_2 \tag{5}$$

The simplified form of each expression, for ease of mathematical calculations, is as follows:

$v_1 = x(r_2) - x(r_1)$

$v_2 = x(r_2) + x(r_1) - \frac{2}{\ell_2} \sum_{r_1+1}^{r_2-1} x(j)$

$\ell_1 = r_2 - r_1$,$\ell_2 = \ell_1 - 1$,$\ell_3 = \ell_1 + 1$

3. Main results

In this section of the paper, we examine the reduction of conservatism in discrete-time delayed switching dynamical systems using the Nonmonotonic Lyapunov-Krasovskii approach. The linear matrix inequalities (LMI) are derived by applying the Lyapunov-Krasovskii function.

3.1. *Non-Monotonic Lyapunov–Krasovskii Stability Analysis Method*

In the non-monotonic Lyapunov stability framework, the Lyapunov function may exhibit temporary increases, provided it decreases over time. The m-step difference of the function is defined as follows:

$$\Delta_m V = V(x_{(k+m)}) - V(x_{(k)}) \tag{6}$$

where m is the number of steps over which the Lyapunov-Krasovskii function is allowed to increase.

In discrete-time systems with delays, the delay can be considered in several forms:

1. Known constant
2. Unknown constant
3. Bounded time-varying

While time-varying delays constitute the most general scenario, many industrial systems, including process control applications, typically experience constant delays [26]. The proposed theories in this paper are capable of analyzing the stability of systems with delays in the "known constant" or "unknown constant" cases.

Consider the state-space model of a discrete-time system subject to a constant but unknown delay, described as follows:

$$\begin{cases} x_{(k+1)} = S_{\sigma(k)} x_{(k)} + S_{d\sigma(k)} x_{(k-d)} \\ x(\theta) = \varphi(\theta), \quad \theta = -d, -d+1 \cdots\cdots 0 \end{cases} \quad (7)$$

where $S_{\sigma(k)}$ and $S_{d\sigma(k)}$ are elements of $R^{n\times n}$, $\varphi(\theta)$ represents the initial conditions, d represents the maximum value of the unknown delay, and takes a positive integer value. Moreover, where V_1, V_2, V_3 are given by equations (9) to (11).

$$V(k) = V_1(k) + V_2(k) + V_3(k) < 0 \quad (8)$$

$$V_1(k) = x_{(k)}^T \mathrm{B}\, x_{(k)} \quad (9)$$

$$+ \quad V_2(k) = \sum_{j=k-d}^{k-1} x_{(j)}^T F\, x_{(j)} \quad (10)$$

$$+ \quad V_3(k) = \ell_2 \ell_1 \sum_{\theta=-d}^{-1} . \sum_{j=k+\theta}^{k-1} \eta_{(j)}^T H\, \eta_{(j)} \quad (11)$$

where $\ell_1 = d-1$, $\ell_2 = \ell_1 - 1$ and $\eta_{(j)} = x_{(j+m)} - x_{(j)}$. We now proceed to present the primary results of this study.

Theorem 3: Assume the discrete-time system (7) is subject to a constant, unknown delay and specified initial conditions. Then, the system is m-step NMLK stable if $1 \le m < d$ and the matrices $B \,\epsilon\, R^{2n\times 2n}$, $F \,\epsilon\, R^{n\times n}$, $H \,\epsilon\, R^{n\times n}$ are positive definite, such that:

$$\mathfrak{I}_1 + \mathfrak{I}_2 + \mathfrak{I}_3 < 0 \quad (12)$$

In which the parameters are defined as follows. These expressions construct the functional, which must ultimately decrease in sum:

$$\mathfrak{I}_1 = \Psi_1^T B \Psi_1 - \Psi_2^T B \Psi_2 \quad (13)$$

$$\mathfrak{I}_2 = \sum_{j=0}^{m-1} ((\text{Å}_j^1)^T F \text{Å}_j^1 - (\text{Å}_j^2)^T F \text{Å}_j^2) \quad (14)$$

$$\mathfrak{I}_3 = \sum_{j=0}^{m-1} (d-m+j+1)(\text{Å}_{j+1}^1 - \text{Å}_j^1)^T H (\text{Å}_{j+1}^1 - \text{Å}_j^1) - \sum_{j=0}^{m-2} (j+1)(\text{Å}_{j+1}^2 - \text{Å}_j^2)^T H (\text{Å}_{j+1}^2 - \text{Å}_j^2) - \frac{m}{\ell_1} (\varphi_m)^T H (\varphi_m) - \frac{3m\ell_2}{\ell_1 \ell_3} (\Pi_m)^T H (\Pi_m) \quad (15)$$

Based on the provided definition, when encountering complex computational expressions, our main objective is to reduce their complexity and simplify them the greatest extent possible. To this end, we employ the following expressions to represent longer terms in a concise and substitutable form.

$$\eta_{(j)} = x_{(j+m)} - x_{(j)} \quad (16)$$

$$\Psi_1 = \begin{bmatrix} \text{Å}_m^1 \\ \mathcal{g}_{m-2} - \text{Å}_{m-1}^2 + \sum_{j=0}^{m-1} L_j^1 \end{bmatrix} \quad (17)$$

$$\Psi_2 = \begin{bmatrix} \mathcal{g}_1 \\ \mathcal{g}_{m+2} + \sum_{j=0}^{m-2} L_j^2 \end{bmatrix}$$

$$\text{Å}_0^1 = \mathcal{g}_1$$

$$\text{Å}_j^1 = \mathrm{S}_i \text{Å}_{j-1}^1 + S_{di} \text{Å}_{j-1+(1)}^1$$

$$\text{Å}_0^2 = \mathcal{g}_2$$

$$\text{Å}_j^2 = \mathrm{S}_i \text{Å}_{j-1}^2 + S_{di} \text{Å}_{j-1+(1)}^2$$

$$\varphi_m = (\mathcal{g}_1 - L_{m-1}^2)$$

$$\Pi_m = \mathcal{g}_1 + \frac{\ell_2 + 2}{\ell_2} L_{m-1}^2 - \frac{2}{\ell_1} \mathcal{g}_{m+2}$$

$$\xi_{(x)}^T = \left\{ x_{(k)} \text{ ‘} x_{(k-d)} \text{‘ } x_{(k-d+1)} \text{ ‘} \cdots \therefore x_{(k-d+m-1)} \text{‘ } \sum_{j=k-d+m}^{k-1} x(j) \right\}$$

Proof: In the stability analysis process, each of the terms $\Delta_m V = \Delta_m V_1 + \Delta_m V_2 + \Delta_m V_3$ is calculated in the following steps:

Step 1: Calculation of $\Delta_m V_1$:

By utilizing the state-space representation provided in (7) along with the definitions introduced, the stability analysis will proceed; that is, the second difference of the first Lyapunov function in equation (9) is calculated to establish a foundation for a precise examination of the system's dynamic behavior.

The block row vectors $\mathcal{g}_i$, for i=1,2,…,(m+2) are defined as follows:

$$\mathcal{g}_i = [\, 0_{n\times(i-1)n} \;\; . \;\; I_{n\times n} \, . \;\; 0_{n\times(4-i)n} \,] \quad (18)$$

$$\Delta_m V_1 = x_{(k+m)}^T \mathrm{B} x_{(k+m)} - x_{(k)}^T \mathrm{B} x_{(k)} \quad (19)$$

First, the state-space representation according to the system dynamics defined in equation (7) is written:

$$\Delta_m V_1 = \begin{bmatrix} x_{(k+m)} \\ \sum_{j=k-d}^{k-1} x^T(j+m) \end{bmatrix}^T B \begin{bmatrix} x_{(k+m)} \\ \sum_{j=k-d}^{k-1} x^T(j+m) \end{bmatrix} - \begin{bmatrix} x_{(k)} \\ \sum_{j=k-d}^{k-1} x^T(j) \end{bmatrix}^T B \begin{bmatrix} x_{(k)} \\ \sum_{j=k-d}^{k-1} x^T(j) \end{bmatrix} \quad (20)$$

Consider the following equations: two commonly used expansions in Lyapunov-Krasovskii functions are expressed as follows. These relations have been developed and rewritten to provide a deeper understanding of the mathematical foundations presented in this paper. By substituting m as the desired time step, it can be observed that many complex expressions, which contribute to increasing system conservatism, are eliminated and simplified. As a result, a more optimized form of the equations is obtained. The modified relations are presented as follows:

$$\sum_{j=k-d}^{k-1} x(j+m) = \sum_{j=k-d+m-1}^{k-1} x(j) - x(k-d+m-1) + \sum_{j=0}^{m-1} x(k+j) \tag{21}$$

$$\sum_{j=k-d}^{k-1} x(j) = \sum_{j=k-d+m-1}^{k-1} x(j) + \sum_{j=-d}^{-d+m-2} x(k+j) \tag{22}$$

Considering the defined expressions and substituting the interval bounds, and also utilizing the relations provided in equations (21) and (22), the general m-step form of $\Delta_m V_1$ is obtained. This analytical expression provides a solid foundation for the subsequent steps of the proof and analysis, and allows for a precise examination of stability within the multi-step framework.

$$\Delta_m V_1 = \xi_{(x)}^T \left(\begin{bmatrix} Å_m^1 \\ \mathcal{g}_{m+2} - Å_{m-1}^2 + \sum_{j=0}^{m-1} Å_j^1 \end{bmatrix}^T \times B \begin{bmatrix} Å_m^1 \\ \mathcal{g}_{m+2} - Å_{m-1}^2 + \sum_{j=0}^{m-1} Å_j^1 \end{bmatrix} - \begin{bmatrix} \mathcal{g}_1 \\ \mathcal{g}_{m+2} + \sum_{j=0}^{m-2} Å_j^2 \end{bmatrix}^T \times B \begin{bmatrix} \mathcal{g}_1 \\ \mathcal{g}_{m+2} + \sum_{j=0}^{m-2} Å_j^2 \end{bmatrix} \right) \xi(x) \tag{23}$$

Step 2: Calculation of $\Delta_m V_2$:

$$\Delta_m V_2 = \sum_{j=k-d}^{k-1} x_{(j+m)}^T F x_{(j+m)} - \sum_{j=k-d}^{k-1} x_{(j)}^T F x_{(j)} \tag{24a}$$

By substituting the interval limits into equation (25) and considering equations (22) and (23), the following fundamental structures will be reached. Solving the complex mathematical expressions leads to the elimination of other terms in the algebraic calculations.

$$\Delta_m V_2 = x_{(k+m-1)}^T F x_{(k+m-1)} + \cdots + x_{(k)}^T \mathrm{F} x_{(k)} - x_{(k-d+m-1)}^T F x_{(k-d+m-1)} - \cdots - x_{(k-d)}^T F x_{(k-d)} \tag{24b}$$

By utilizing the defined expressions to simplify complex mathematical terms and substituting them into equation (17), the following relation is derived, which provides the foundation for the subsequent analysis and computations.

$$\Delta_m V_2 = \xi_{(x)}^T \left\{ \sum_{j=0}^{m-1} ((Å_j^1)^T F Å_j^1 - (Å_j^2)^T F Å_j^2) \right\} \xi(x) \tag{25}$$

Step 3: Calculation of $\Delta_m V_3$:

$$\Delta_m V_3 = \ell_2 \ell_1 \left(\sum_{\theta=-d}^{-1} \cdot \sum_{j=k+\theta}^{k-1} \eta_{(j+m)}^T H \eta_{(j+m)} - \sum_{\theta=-d}^{-1} \cdot \sum_{j=k+\theta}^{k-1} \eta_{(j)}^T H \eta_{(j)} \right) \tag{26a}$$

By substituting the limit values into the relevant equations and reapplying relations (21) and (22) to simplify the intermediate steps, the following analytical expression is ultimately obtained, which serves as the basis for the subsequent steps of the proof.

$$\Delta_m V_3 = \ell_2 \ell_1 d (\eta_{(k)}^T H \eta_{(k)} + \cdots + \eta_{(k+m-1)}^T H \eta_{(k+m-1)} - \sum_{j=k-d}^{k-1} \eta_{(j)}^T H \eta_{(j)} - \sum_{j=k-d+1}^{k} \eta_{(j)}^T H \eta_{(j)} - \cdots - \sum_{j=k-d+m-1}^{k+m-2} \eta_{(j)}^T H \eta_{(j)} \tag{26b}$$

$$\Delta_m V_3 = \xi_{(x)}^T \left(\ell_2 \ell_1 \sum_{j=0}^{m-1} (d-m+1+j)(Å_{j+1}^1 - Å_j^1)^T \times H (Å_{j+1}^1 - Å_j^1) - \sum_{j=0}^{m-2} (j+1)(Å_{j+1}^2 - Å_j^2)^T \times H (Å_{j+1}^2 - Å_j^2) \right) \xi(x) + \Gamma \tag{26c}$$

The summation term Γ can be easily eliminated since it is definitively negative. However, doing so may increase conservatism. To mitigate this effect, instead of eliminating the term, an upper bound is computed, which serves to reduce conservatism. For this purpose, Lemma 1 (Abel's lemma) is applied and extended to the m-step case to further reduce the system's conservatism.

$$\Gamma = -m \sum_{j=k-d+m-1}^{k-1} \eta_{(j)}^T H \eta_{(j)} \tag{27}$$

Finally, we obtain:

$$-m\sum_{j=k-d+m-1}^{k-1}\eta_{(j)}^{T}H\,\eta_{(j)} \le -\frac{m}{\ell_1}\Big(x_{(k)} - x_{(k-d+m-1)}\big)\big)^{T}H\big(x_{(k)} - x_{(k-d+m-1)}\big)\Big) - \frac{3m\ell_2}{\ell_1\ell_3}\Bigg(x_{(k)} + x_{(k-d+m-1)} - \frac{2}{\ell_2}\bigg(-x_{(k-d+m-1)} + \sum_{j=k-d+m-1}^{k-1} x_{(j)}\bigg)\Bigg)^{T} H\Bigg(x_{(k)} + x_{(k-d+m-1)} - \frac{2}{\ell_2}\bigg(-x_{(k-d+m-1)} + \sum_{j=k-d+m-1}^{k-1} x_{(j)}\bigg)\Bigg) \quad (28)$$

3.2. Robust Non-Monotonic Lyapunov–Krasovskii Stability Analysis Method

In practical scenarios, most systems expressed through mathematical models inevitably involve some degree of uncertainty. Consequently, incorporating such uncertainties into the stability analysis framework is a key requirement for reliable control system design. This section introduces the robust non-monotonic Lyapunov–Krasovskii (RNMLK) approach and provides the corresponding stability results for uncertain time-delay systems, as formulated in equation (29).

$$\begin{cases} x_{(k+1)} = (S + \delta S(k))_{\sigma(k)} x_{(k)} + (S + \delta S(k))_{\sigma(k)d}\, x_{(k-d)} \\ x(\theta) = \varphi(\theta)\,, \quad \theta = -d\,, -d+1 \cdots\cdots 0 \end{cases} \quad (29)$$

where $\varphi(\theta)$ represents the initial conditions and $\delta S_i(k)$ and $\delta S_{id}(k)$ are the uncertainty matrices, defined as follows:

$$[\delta S_i(k) \quad \delta S_{id}(k)] = QZ(k)[M_a \quad M_d] \quad (30)$$

where Q, M_a and M_d are constant matrices and $Z(k)$ is the uncertainty matrix satisfying the following inequality:

$$Z(k)^{T}Z(k) \le I \quad (31)$$

Theorem 4: The robust non-monotonic Lyapunov–Krasovskii (RNMLK) theorem with a specified step states that, in this case, the number of steps is m=2.

Theorem 4 The discrete-time linear system with uncertainty and time delay, equation (29), possesses two-step robust non-monotonic Lyapunov–Krasovskii (2-step RNMLK) stability if there exist positive definite matrices $B \epsilon R^{2n\times 2n}$, $F \epsilon R^{n\times n}$, $H \epsilon R^{n\times n}$, together with scalars $\varepsilon_i > 0$, $i = 1 \ldots 5$, such that:

$$\mathfrak{I}_{11} + \begin{bmatrix} 0_{4n\times 4n} & [\mathfrak{I}_{12}^{T}] \\ [\mathfrak{I}_{12}] & \mathfrak{I}_{22} \end{bmatrix} < 0 \quad (32)$$

(33)

$$\mathfrak{I}_{12}^{T} = \Big[\varphi_2^{T}HQ, \Pi_2^{T}HQ, (\mathring{A}_1^2)^{T}FQ \ldots (\mathring{A}_1^2 - (\mathring{A}_0^2)^{T}HQ, (\mathring{A}_1^1)^{T}, \sqrt{d-1}\,\big(\bar{\mathring{A}}_2 - \mathring{A}_1^1\big)^{T} \ldots\ldots \sqrt{d}\,\big(\mathring{A}_1^1 - \mathring{A}_0^1\big)^{T}, \bar{\Psi}_1^{T}\Big]$$

$$\mathfrak{I}_{22} = \begin{bmatrix} 0_{4n\times 4n} & 0_{4n\times n} & 0_{4n\times n} & 0_{4n\times n} & [0_{4n\times n} \quad 0_{4n\times n}] \\ 0_{n\times 4n} & \lambda & -\sqrt{d-1}\,\lambda & \sqrt{d}\,\lambda & [0_{n\times n} \quad \lambda] \\ 0_{n\times 4n} & -\sqrt{d-1}\,\lambda & (d-1)\lambda & -\sqrt{d(d-1)}\lambda & [0_{n\times n} \quad -\sqrt{d-1}\,\lambda] \\ 0_{n\times 4n} & \sqrt{d}\,\lambda & -\sqrt{d(d-1)}\lambda & d\lambda & [0_{n\times n} \quad \sqrt{d}\,\lambda] \\ \begin{bmatrix} 0_{n\times 4n} \\ 0_{n\times 4n} \end{bmatrix} & \begin{bmatrix} 0_{n\times n} \\ \lambda \end{bmatrix} & \begin{bmatrix} 0_{n\times n} \\ -\sqrt{d-1}\,\lambda \end{bmatrix} & \begin{bmatrix} 0_{n\times n} \\ \sqrt{d}\,\lambda \end{bmatrix} & \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \lambda \end{bmatrix} \end{bmatrix} \quad (34a)$$

$$\mathfrak{I}_{11} = diag\{(\omega + \bar{\omega} + \varepsilon_5^{-1}(M_a g_1 + M_d g_2)^{T}(M_a g_1 + M_d g_2)), -\tfrac{2}{(d-1)}(Q^{T}HQ - \varepsilon_1), -\tfrac{6(d-2)}{d(d-1)} \times (Q^{T}HQ - \varepsilon_2), -(Q^{T}HQ - \varepsilon_3), -(Q^{T}HQ - \varepsilon_4) - F^{-1}, -H^{-1}, -H^{-1}, -B^{-1}\} \quad (34b)$$

$$\bar{\omega} = -\Big(\tfrac{2}{(d-1)}\varepsilon_1 + \tfrac{6d}{(d-2)(d-1)}\varepsilon_2 + \varepsilon_3 + \varepsilon_4\Big)J_1^{T}J_1 - \tfrac{2}{(d-1)}\varphi_2^{T}H\varphi_2 - \tfrac{6(d-2)}{d(d-1)}\Pi_2^{T}H\Pi_2 - (\mathring{A}_1^2)^{T}F\mathring{A}_1^2 - \big(\mathring{A}_1^2 - \mathring{A}_0^2\big)^{T}H\big(\mathring{A}_1^2 - \mathring{A}_0^2\big) \quad (34c)$$

$$\omega = (\mathring{A}_0^1)^{T}F\,\mathring{A}_0^1 - \Psi_2^{T}B\Psi_2 - (\mathring{A}_0^2)^{T}F\mathring{A}_0^2 \quad (34d)$$

$$\bar{\Psi}_1 = \begin{bmatrix} \bar{\mathring{A}}_2^1 \\ \mathring{A}_0^1 + \mathring{A}_1^1 + \bar{\mathring{A}}_1^2 + g_4 \end{bmatrix} \quad (34e)$$

$$\Psi_2 = \begin{bmatrix} g_1 \\ (\mathring{A}_0^2 + g_4) \end{bmatrix}$$

$$\mathring{A}_0^1 = g_1$$

$$\mathring{A}_1^1 = S_i\, g_1 + S_{id}\, g_2$$

$$\mathring{A}_0^2 = g_2$$

$$\mathring{A}_1^2 = S_i\, g_2 + S_{id}\, g_3$$

where:

$$\bar{\mathring{A}}_2^1 = (\|S_i\|^2 + 2\|S_i\|\|Q\|\|M_a\| + \|Q\|^2\|M_a\|^2)g_1 + 2(\|S_iS_{id}\| + \|S_i\|\|Q\|\|M_a\| + \|Q\|\|M_a\|\|S_{id}\| + \|Q\|\|M_a\|\|Q\|\|M_d\|)g_2 + (\|S_{di}\|^2 + 2\|S_{id}\|\|Q\|\|M_d\| + \|Q\|^2\|M_d\|^2)g_3 \quad (34f)$$

$$\begin{aligned}\bar{\AA}_1^2 &= (\|S_i\| + \|S_i\|\|Q\|\|M_a\| \\ &+ \|Q\|\|M_a\|\|S_{id}\| \\ &+ \|Q\|\|M_a\|\|Q\|\|M_d\|)\mathscr{g}_2 + (\|S_{id}\|^2 \\ &+ 2\|S_{di}\|\|Q\|\|M_d\| + \|Q\|^2\|M_d\|^2)\mathscr{g}_3\end{aligned} \tag{34g}$$

$$\begin{aligned}&\varphi_2 = \left(\AA_0^1 - \AA_1^2\right) \\ &\Pi_2 = \AA_0^1 + \frac{d}{(d-2)}\AA_1^2 - \frac{2}{d-2}\mathscr{g}_4 \\ &J_1 = M_a\mathscr{g}_2\,, \lambda = \varepsilon_5 QQ^T\end{aligned} \tag{34h}$$

Proof. By taking into account the uncertain coefficient matrices and employing a method similar to that presented in Theorem 3, the two-step stability criterion for the uncertain time-delay system described in equation (29) is derived. This criterion enables a precise analysis of the system's stability under the influence of uncertainties and provides a solid foundation for the subsequent steps of the proof and examination of the system's dynamic.

$$\tilde{S}_i = S_i + \delta S_i(k) \quad , \tilde{S}_{id} = S_{id} + \delta S_{id}(k) \tag{35}$$

where:

$$\tilde{\mathfrak{I}}_1 + \tilde{\mathfrak{I}}_2 + \tilde{\mathfrak{I}}_3 < 0 \tag{36}$$

$$\tilde{\mathfrak{I}}_1 = \tilde{\Psi}_1^T B \tilde{\Psi}_1 - \Psi_2^T B \Psi_2 \tag{37a}$$

$$\begin{aligned}\tilde{\mathfrak{I}}_2 = (\tilde{\AA}_1^1)^T F \tilde{\AA}_1^1 + (\AA_0^1)^T F \AA_0^1 \\ - (\AA_0^2)^T F \AA_0^2 \\ + (\tilde{\AA}_1^2)^T F \tilde{\AA}_1^2\end{aligned} \tag{37b}$$

$$\begin{aligned}\tilde{\mathfrak{I}}_3 = \ell_2\ell_1(d-1)\left[\tilde{\AA}_1^1 - \AA_0^1\right]^T H \Big[\tilde{\AA}_1^1 \\ - \AA_0^1\Big] + d\left[\tilde{\AA}_2^1 - \tilde{\AA}_1^1\right]^T \\ - H\left[\tilde{\AA}_2^1 - \tilde{\AA}_1^1\right] \\ - \left[\tilde{\AA}_1^2 - \AA_0^2\right]^T H \Big[\tilde{\AA}_1^2 \\ - \AA_0^2\Big] - \frac{2}{d-1}\tilde{\varphi}_2{}^T H \tilde{\varphi}_2 \\ + \frac{6(d-2)}{d(d-1)}\tilde{\Pi}^T{}_2 H \tilde{\Pi}_2\end{aligned} \tag{37c}$$

and

$$\begin{aligned}&\tilde{\Psi}_1 = \begin{bmatrix} \tilde{\AA}_2^1 \\ \AA_0^1 + \AA_1^1 + \tilde{\AA}_1^2 + \mathscr{g}_4 \end{bmatrix} \\ &\Psi_2 = \begin{bmatrix} \mathscr{g}_1 \\ \AA(_0^2 + \mathscr{g}_4) \end{bmatrix} \\ &\AA_0^1 = \mathscr{g}_1 \\ &\tilde{\AA}_1^1 = \tilde{S}_i\,\mathscr{g}_1 + \tilde{S}_{id}\,\mathscr{g}_2 \\ &\AA_0^2 = \mathscr{g}_2 \\ &\tilde{\AA}_1^2 = \tilde{S}_i\,\mathscr{g}_2 + \tilde{S}_{id}\,\mathscr{g}_3 \\ &\tilde{\AA}_2^1 = \tilde{S}_i^2\,\mathscr{g}_1 + \left(\tilde{S}_i\tilde{S}_{id} + \tilde{S}_{id}\tilde{S}_i\right)\mathscr{g}_2 + \tilde{S}_{id}^2\mathscr{g}_3 \\ &\tilde{\varphi}_2 = \left(\AA_0^1 - \tilde{\AA}_1^2\right). \\ &\tilde{\Pi}_2 = \AA_0^1 + \frac{d}{(d-2)}\tilde{\AA}_1^2 - \frac{2}{d-2}\mathscr{g}_4\end{aligned}$$

Given the aforementioned conditions, the matrices marked with a tilde ~ include the uncertain coefficient matrices $\tilde{S}_i$ or $\tilde{S}_{id}$. Accordingly, the uncertain parts include $\tilde{\Psi}^T{}_1 B \tilde{\Psi}_1$, The parameters are defined as follows:

$$\begin{aligned}\delta\omega = -(\tilde{\AA}_1^2)^T F \tilde{\AA}_1^2 - \left(\tilde{\AA}_1^2 - \AA_0^2\right)^T H \left(\tilde{\AA}_1^2 - \AA_0^2\right) \\ - \frac{2}{(d-1)}\varphi_2^T H \varphi_2 \\ - \frac{6(d-2)}{d(d-1)}\Pi_2^T H \Pi_2\end{aligned} \tag{38}$$

The remaining parameters include $\tilde{\AA}_1^2, \tilde{\AA}_2^1, \tilde{\AA}_1^1$. To derive the robust stability conditions, an upper bound for the uncertain parts is first calculated, following the procedure outlined in Appendix A of [33]. Accordingly, by applying relations (A15), (A16), (A20), and (A32) from Appendix A, inequality (36) is reformulated as follows:

$$\begin{aligned}\tilde{\mathfrak{I}}_1 + \tilde{\mathfrak{I}}_2 + \tilde{\mathfrak{I}}_3 < (d-1)\left[\tilde{\AA}_1^1 - \AA_0^1\right]^T H \Big[\tilde{\AA}_1^1 \\ - \AA_0^1\Big] + \omega + \varpi + \tilde{\Psi}^T{}_1 B \tilde{\Psi}_1 \\ + (\tilde{\AA}_1^1)^T F \tilde{\AA}_1^1 + d\left[\bar{\AA}_2^1 - \tilde{\AA}_1^1\right]^T \\ - H\left[\bar{\AA}_2^1 - \tilde{\AA}_1^1\right]\end{aligned} \tag{39}$$

where ω, ϖ and $\tilde{\Psi}_1$ defined in equations (34-c)–(34-e). Applying the Schur complement lemma:

$$\tilde{\mathfrak{I}}_1 + \tilde{\mathfrak{I}}_2 + \tilde{\mathfrak{I}}_3 \le \begin{bmatrix} \omega + \varpi & (\tilde{\AA}_1^1)^T & ꟻ & \mho & \tilde{\Psi}^T{}_1 \\ * & -F^{-1} & 0 & 0 & 0 \\ * & * & -H^{-1} & 0 & 0 \\ * & * & * & -H^{-1} & 0 \\ * & * & * & * & -B^{-1} \end{bmatrix} \tag{40a}$$

In equation (40a), the parameters ꟻ , ℧ are defined as follows:

$$\begin{aligned}&ꟻ = \sqrt{d-1}\left(\tilde{\AA}_2^1 - \tilde{\AA}_1^1\right)^T \\ &\mho = \sqrt{d}(\tilde{\AA}_1^1 - \AA_0^1)^T\end{aligned} \tag{40b}$$

Now, by substituting $\tilde{S}_i = S_i + \delta S_i(k)$, $\tilde{S}_{id} = S_{id} + \delta S_{di}(k)$ into $\tilde{\AA}_1^1$, we obtain $\tilde{\AA}_1^1 = \left(S_i + \delta S_i(k)\right)\mathscr{g}_1 + \left(S_{di} + \delta S_{id}(k)\right)\mathscr{g}_2$By rearranging, $\tilde{\AA}_1^1$ can be rewritten as follows:

$$\tilde{\AA}_1^1 = \AA_1^1 + \delta\AA_1^1 \tag{41}$$

In which $\AA_1^1 = S_i\mathscr{g}_1 + S_{id}\mathscr{g}_2$ and $\delta\AA_1^1 = \delta S_i(k)\mathscr{g}_1 + \delta S_{id}(k)\mathscr{g}_2$, represent the nominal and uncertain parts, respectively. Accordingly, using relation (41), the term $\tilde{\mathfrak{I}}_1 + \tilde{\mathfrak{I}}_2 + \tilde{\mathfrak{I}}_3$ in equation (40a) can be separated as follows:

$$\tilde{\mathfrak{I}}_1 + \tilde{\mathfrak{I}}_2 + \tilde{\mathfrak{I}}_3 \le \wp + \delta\wp \tag{42}$$

where:

$$\delta\wp = \begin{bmatrix} 0 & \delta\AA_1^{1\,T} & \mathcal{T}\delta\AA_1^{1\,T} & \mathcal{K}\delta\AA_1^{1\,T} & \begin{bmatrix}0 & \delta\AA_1^{1\,T}\end{bmatrix} \\ \delta\AA_1^1 & 0 & 0 & 0 & \begin{bmatrix}0 & 0\end{bmatrix} \\ \mathcal{T}\delta\AA_1^1 & 0 & 0 & 0 & \begin{bmatrix}0 & 0\end{bmatrix} \\ \mathcal{K}\delta\AA_1^1 & 0 & 0 & 0 & \begin{bmatrix}0 & 0\end{bmatrix} \\ \begin{bmatrix}0 \\ \delta\AA_1^1\end{bmatrix} & \begin{bmatrix}0 \\ 0\end{bmatrix} & \begin{bmatrix}0 \\ 0\end{bmatrix} & \begin{bmatrix}0 \\ 0\end{bmatrix} & \begin{bmatrix}0 & 0 \\ 0 & 0\end{bmatrix} \end{bmatrix} \tag{43a}$$

$$\wp = \begin{bmatrix} \omega + \bar{\omega} & \left(\mathring{A}_1^1\right)^T & \beth & \mathfrak{S} & \tilde{\Psi}^T{}_1 \\ * & -F^{-1} & 0 & 0 & 0 \\ * & * & -H^{-1} & 0 & 0 \\ * & * & * & -H^{-1} & 0 \\ * & * & * & * & -B^{-1} \end{bmatrix} \tag{43b}$$

In equation (43a-43b), the parameters $\beth, \mathfrak{S}, \mathcal{J}, \mathcal{K}$ are defined as follows:

$$\beth = \sqrt{d-1}\left(\bar{\mathring{A}}_2^1 - \mathring{A}_1^1\right)^T, \mathcal{J} = -\sqrt{d-1} \tag{43c}$$
$$\mathfrak{S} = \sqrt{d}\left(\mathring{A}_1^1 - \mathring{A}_0^1\right)^T, \mathcal{K} = \sqrt{d}$$

Therefore, inequality (36) holds if, for each:

$$\zeta_1 \epsilon R^{4n}, \zeta_i \epsilon R^n, i = 2. \ldots .6 \ .\bar{\zeta} = [\zeta_1^T . \zeta_2^T . \zeta_3^T . \zeta_4^T . \zeta_5^T . \zeta_6^T], \zeta_i \neq 0$$

we have:

$$\bar{\zeta}^T (\wp + \delta\wp)\bar{\zeta} = \bar{\zeta}^T \wp \bar{\zeta} + \bar{\zeta}^T \delta\wp \bar{\zeta} < 0 \tag{44}$$

Then:

$$\bar{\zeta}^T \wp \bar{\zeta} < -\bar{\zeta}^T \delta\wp \bar{\zeta} \tag{45}$$

Considering $\delta\wp$ from equation (43-A), we have:

$$\bar{\zeta}^T \delta\wp \bar{\zeta} = -2\left(\zeta_1^T \delta\mathring{A}_1^{1^T} \zeta_2 - \sqrt{d-1}\delta\mathring{A}_1^{1^T} \delta\mathring{A}_1^{1^T} \zeta_3 + \sqrt{d}\zeta_1^T \delta\mathring{A}_1^{1^T} \zeta_4 + \zeta_1^T \delta\mathring{A}_1^{1^T} \zeta_6\right) \tag{46}$$

By substituting $\delta\mathring{A}_1^1$ from equation (41) and using relations (30), (45), and (46), it follows that:

$$\bar{\zeta}^T \wp \bar{\zeta} < -2max\{\zeta_1^T (M_a \mathscr{g}_1 + M_d e_2)^T (Z^T(k) Q^T \zeta_2 - \sqrt{d-1} Z^T(k) Q^T \zeta_3 + \sqrt{d} Z^T(k) Q^T \zeta_4 + Z^T(k) Q^T \zeta_6) | Z^T(k) Z(k) \leq I\} \leq 0 \tag{47}$$

Its equivalent form is:

$$(\bar{\zeta}^T \wp \bar{\zeta})^2 > 4max\left\{[\zeta_1^T (M_a \mathscr{g}_1 + M_d \mathscr{g}_2)^T (Z^T(k) Q^T \zeta_2 - \sqrt{d-1} Z^T(k) Q^T \zeta_3 + \sqrt{d} Z^T(k) Q^T \zeta_4 + Z^T(k) Q^T \zeta_6]^2 | Z^T(k) Z(k) \leq I\right\} \tag{48}$$

By applying Lemma A.2 in reference [27], the inequality below is satisfied:

$$(\bar{\zeta}^T \wp \bar{\zeta})^2 > 4[\zeta_1^T (M_a \mathscr{g}_1 + M_d \mathscr{g}_2)^T (M_a \mathscr{g}_1 + M_d \mathscr{g}_2)\zeta_1] \times [(Z^T(k) Q^T \zeta_2 - \sqrt{d-1} Z^T(k) Q^T \zeta_3 + \sqrt{d} Z^T(k) Q^T \zeta_4 + Z^T(k) Q^T \zeta_6)^T (Z^T(k) Q^T \zeta_2 - \sqrt{d-1} Z^T(k) Q^T \zeta_3 + \sqrt{d} Z^T(k) Q^T \zeta_4 + Z^T(k) Q^T \zeta_6) | Z^T(k) Z(k) \leq I] \tag{49}$$

In the above inequality, considering that $Z^T(k) Z(k) \leq I$, we conclude that:

$$(\bar{\zeta}^T \wp \bar{\zeta})^2 - 4[\zeta_1]^T [(\mathfrak{A})^T (\mathfrak{A})] [\zeta_1] \tag{50a}$$
$$\begin{bmatrix} \zeta_2 \\ \zeta_3 \\ \zeta_4 \\ \zeta_6 \end{bmatrix}^T \begin{bmatrix} QQ^T & \mathcal{J}QQ^T & \mathcal{K}QQ^T & QQ^T \\ \mathcal{J}QQ^T & (d-1)QQ^T & \mathcal{Y}QQ^T & \mathcal{J}QQ^T \\ \mathcal{K}QQ^T & \mathcal{Y}QQ^T & dQQ^T & \mathcal{K}QQ^T \\ QQ^T & \mathcal{J}QQ^T & \mathcal{K}QQ^T & QQ^T \end{bmatrix} \begin{bmatrix} \zeta_2 \\ \zeta_3 \\ \zeta_4 \\ \zeta_6 \end{bmatrix} > 0$$

In equation (50a), the parameters $\mathcal{Y}.\mathcal{J}.\mathcal{K}.\mathfrak{A}$ are defined as follows:

$$\mathcal{Y} = -\sqrt{d(d-1)}, \mathcal{J} = -\sqrt{d-1} \tag{50b}$$
$$\mathcal{K} = \sqrt{d}, \mathfrak{A} = M_a \mathscr{g}_1 + M_d \mathscr{g}_2$$

Since some parameters are identical to those in equation (50b), they are not repeated, and thus equation (51) is obtained.

By applying Lemma A.3 in reference [27], there exists a positive scalar $\varepsilon_5 > 0$ such that:

$$\wp + \varepsilon_5^{-1} \begin{bmatrix} (\mathfrak{A})^T (\mathfrak{A}) & 0_{4n\times 5n} \\ 0_{5n\times 4n} & 0_{5n\times 5n} \end{bmatrix} + \varepsilon_5 \begin{bmatrix} 0_{4n\times 4n} & 0_{4n\times n} & 0_{4n\times n} & 0_{4n\times n} & [0_{4n\times n} \quad 0_{4n\times n}] \\ 0_{n\times 4n} & \lambda & \mathcal{J}\lambda & \mathcal{K}\lambda & [0_{n\times n} \quad \lambda] \\ 0_{n\times 4n} & \mathcal{J}\lambda & (d-1)\lambda & \mathcal{Y}\lambda & [0_{n\times n} \quad \mathcal{J}\lambda] \\ 0_{n\times 4n} & \mathcal{K}\lambda & \mathcal{Y}\lambda & d\lambda & [0_{n\times n} \quad \mathcal{K}\lambda] \\ \begin{bmatrix} 0_{n\times 4n} \\ 0_{n\times 4n} \end{bmatrix} & \begin{bmatrix} 0_{n\times n} \\ \lambda \end{bmatrix} & \begin{bmatrix} 0_{n\times n} \\ \mathcal{J}\lambda \end{bmatrix} & \begin{bmatrix} 0_{n\times n} \\ \mathcal{K}\lambda \end{bmatrix} & \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \lambda \end{bmatrix} \end{bmatrix} < 0 \tag{51}$$

where λ is defined in equation (34h). Then, by performing some transformations, equation (33) is obtained, completing the proof.

The robust stability condition presented in equation (33) involves $\varepsilon_5^{-1}, B^{-1}, F^{-1}, H^{-1}$, which makes it nonlinear. Consequently, the resulting robust stability condition is not a linear matrix inequality (LMI). Therefore, the problem is solved using *Algorithm I*, thereby converting the non-convex bilinear matrix inequality (BMI) problem into a nonlinear optimization problem formulated with LMIs [28]. Defining $R_1 = B^{-1}, R_2 = F^{-1}, R_3 = H^{-1}, R_4 = \varepsilon_5^{-1}$ *Algorithm I* can be applied to the corresponding BMI problem (33).

Algorithm I: Solving the BMI Problem

Minimization: subject to $Trace\{BR_1 + FR_2 + HR_3\} + Trace\{\varepsilon_5 R_4\}$ in relation (33) and

$$B > 0, F > 0, H > 0, \varepsilon_i > 0, i = 1 \ldots 5$$
$$\begin{bmatrix} B & I \\ * & R_1 \end{bmatrix}, \begin{bmatrix} F & I \\ * & R_2 \end{bmatrix}, \begin{bmatrix} H & I \\ * & R_3 \end{bmatrix}, \begin{bmatrix} \varepsilon_5 & I \\ * & R_4 \end{bmatrix}$$

The algorithm employs an iterative procedure, in which initial estimates of the decision matrices are updated at each iteration. At every step, the BMI problem is approximated using LMI-based relaxations, enabling numerical solution via standard LMI solvers. The iteration continues until either convergence is reached or the minimization criterion of the objective function

$$Trace\{BR_1 + FR_2 + HR_3\} + Trace\{\varepsilon_5 R_4\}$$

is satisfied. This approach ensures that the original non-convex problem is systematically handled while preserving the robustness and stability guarantees of the proposed design.

4. Numerical example

Three numerical examples are presented to illustrate and assess the effectiveness of the stability analysis criteria proposed in Theorems 1,2,3 and 4. It is worth noting that the results presented in this section were obtained using MATLAB. For performing the computations, the YALMIP toolbox along with the PENLAB solver was employed. Minor differences in the results may occur due to variations in settings and the choice of different solvers .Moreover, in the numerical examples, the delay is considered a known constant, which is consistent with the practical scenarios discussed in the main results and demonstrates the applicability of the proposed NMLK method under this case.

Example 1: Consider the following delayed switching system:

$$x_{(k+1)} = S_{\sigma(k)} x_{(k)} + S_{\sigma(k)d} x_{(k-d)} \quad (52)$$

where the subsystem is defined as $\sigma(k)$: $\sigma(k) = i$

Then, two subsystems are defined for this delayed system $(i = 1, 2)$

In this example, $q = 0.8$ and $\beta = 0$ are considered.

$$(x_{k+1})_{i=1} = \begin{bmatrix} q & 0 \\ (1-q)\cdot 0.25 & q+0.1 \end{bmatrix} x_k + \begin{bmatrix} -(1-q)\cdot 0.5 & \beta \\ -(1-q)\cdot 1.4 & -(1-q)\cdot 0.5 \end{bmatrix} x_{(k-75)}$$

$$(x_{k+1})_{i=2} = \begin{bmatrix} q+0.1 & q\cdot 0.225 \\ -(1-q)\cdot 0.5 & q\cdot 0.5 \end{bmatrix} x_k + \begin{bmatrix} -(1-q)\cdot 0.3 & q\cdot 0.025 \\ -(1-q)\cdot 0.05 & -(1-q)\cdot 0.15 \end{bmatrix} x_{(k-75)}$$

Table I. Maximum upper bound of *d*

		$\boldsymbol{\beta = 0}$	$\boldsymbol{\beta = 0.058}$
[13]		17	-
[3]		18	-
[29]		19	-
[14]			57
Theorem 3	m=1	∞	57
	m=2	∞	57
	⋮	⋮	⋮
	m=57	∞	58

Table I compares the maximum upper bound of the delay ddd obtained using Theorem 3 with different non-monotonicity steps in this study against the stabilization criteria derived from the Lyapunov functional in [13], the finite-sum inequality-based approach in [3], and the use of zero equalities in [29], for various values of β. It can be observed that for β=0, the proposed NMLK algorithm is less conservative than the methods reported in [13], [3], and [29]. In cases where the symbol ∞ is used in the table, this indicates that, for β=0, the system remains stable for any delay at each time step. By employing the NMLK functional and applying Theorem 3, the stability of system (52) is guaranteed. also, the introduced (NMLKF) approach demonstrates superior performance compared to the sum-of-squares inequality–based method in [3] and the Abel lemma–based approach in [14]. Specifically, the method in [3] fails to establish stability when β=0, whereas the Abel-lemma-based method [14] provides a relatively acceptable stability margin for d ≤ 57. However, Theorem 3 outperforms all other methods, including stability analysis under arbitrary switching [30] and discrete-time switched systems containing unstable subsystems [31]. In both cases, the stability problem is formulated in terms of Linear Matrix Inequalities (LMIs) by employing Lyapunov functions. Compared with these approaches, Theorem 3 ensures a wider stability margin while significantly reducing conservatism. The proposed non-monotonic Lyapunov–Krasovskii functional (NMLK) associated with system (52), for β=0, m=57, and the following initial conditions, is illustrated in Fig. 2. $X(0)=[\,10\ \ -10]^T$

Fig.1 illustrates the system's state trajectory, showing that during the stability analysis, the system converges to a constant value and maintains stability despite the delays.

Fig.2 depicts the non-monotonic decreasing nature of the NMLK function, where the function increases in certain intervals, yet its overall trend remains decreasing. The increases observed in Figure 1 are at most equal to the non-monotonic step size of m = 57. Figure 2 also consists of two parts: the first part shows the proposed Lyapunov function plotted over 2000 iterations, while in the second part, a segment of the main graph is zoomed in and the number of iterations is reduced from 2000 to 200, in order to better illustrate the non-monotonicity as well as the switching between subsystems.

Fig.3 presents the switching function, which indicates which subsystem is active at the defined time steps. In this example, the switching function varies randomly throughout the specified number of iterations.

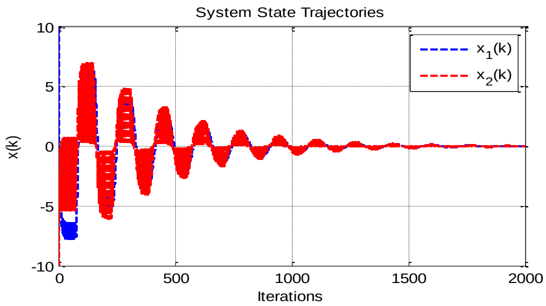

Fig.1. States of the system for Example 1

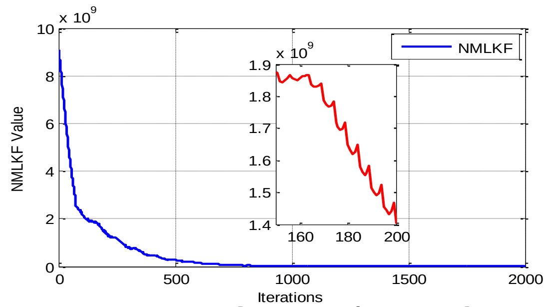

Fig.2. Proposed NMLKF for Example 1

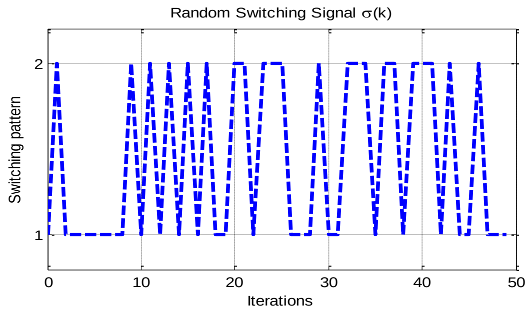

Fig.3. Switching pattern between subsystems

Example 2. Let us examine the following uncertain system:

$$x_{(k+1)} = (S+\delta S(k))_i x_{(k)} + (S+\delta S(k))_{id} x_{(k-8)} \quad (53)$$

If the unknown parameter q belongs to the interval $q \in [0.1]$, then by substituting q =0 we obtain subsystem 1, and by substituting q =1 we obtain subsystem 2.

$$S_i(q) = \begin{bmatrix} -0.85-0.05q & -0.52+0.07q \\ 0.41-0.03q & -0.80+0.14q \end{bmatrix}$$

$$S_{id}(q) = \begin{bmatrix} 0.31-0.03q & 0.10+0.02q \\ -0.20+0.08q & -0.10+0.01q \end{bmatrix}$$

In which considering the uncertainty matrics (30)

$$Q_1 = \begin{bmatrix} 0.06 \\ 0.1 \end{bmatrix}, M_{a1} = [0.15 \quad 0.1], M_{d1} = [0.2 \quad 0.1]$$

$$Q_2 = \begin{bmatrix} 0.35 \\ 0.09 \end{bmatrix}, M_{a2} = [0.16 \quad 0.11], M_{d2} = [0.22 \quad 0.09]$$

In this example, Theorem 4 can still guarantee the stability of the system in the presence of parameter uncertainty, and the proposed method ensures stability in the proof process more effectively compared with other approaches. The objective of this example is to evaluate the performance of the proposed method in the robust stability analysis of time-delay switched systems. The RNMLKF method, in contrast to previous studies, utilizes a non-monotonic Lyapunov–Krasovskii functional with the aim that, if the uncertainty is in the form of a convex polytope, the constraints can be converted into LMIs [32]-[36], or, in the case of norm-bounded uncertainty, as in [34]-[35], the robust stability can be proven for T-S fuzzy models using soft-computing techniques from fuzzy control. Theorem 4 provides a stability margin with reduced conservatism. The initial conditions are given as: $x(0) = [1 \quad 1]^T$

Fig.4 Illustrates the system's state trajectories and indicates that during the stability analysis, the system converges to a constant value and remains stable in the presence of delays.

Fig.5 depicts the non-monotonic decreasing nature of the NMLK function, where the function increases in certain intervals, yet its overall trend remains decreasing. The increases observed in Figure 1 are at most equal to the non-monotonic step size of m = 2. Figure 2 also consists of two parts: the first part shows the proposed Lyapunov function plotted over 2000 iterations, while in the second part, a segment of the main graph is zoomed in and the number of iterations is reduced from 2000 to 200, in order to better illustrate the non-monotonicity as well as the switching between subsystems.

Fig.6 presents the switching function, which indicates which subsystem is active at the defined time steps. In this example, the switching function varies randomly throughout the specified number of iterations.

Fig.7 State norms under parametric uncertainty show bounded L_2 energy and limited peak response L_∞, allowing evaluation of robust stability performance under switching.

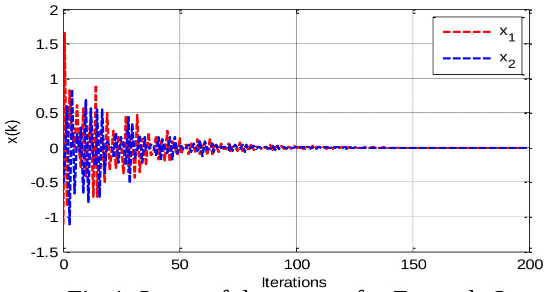

Fig.4. States of the system for Example 2

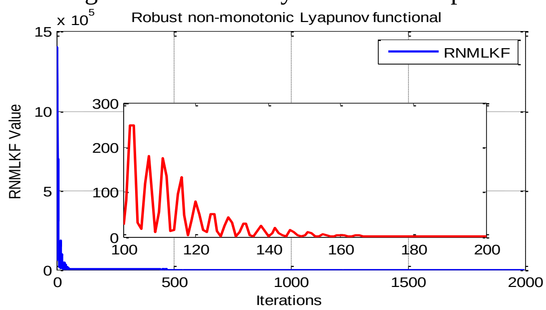

Fig. 5. Proposed RNMLKF for Example 2

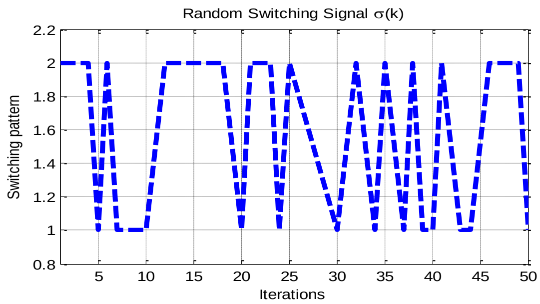

Fig.6. Switching pattern between subsystems

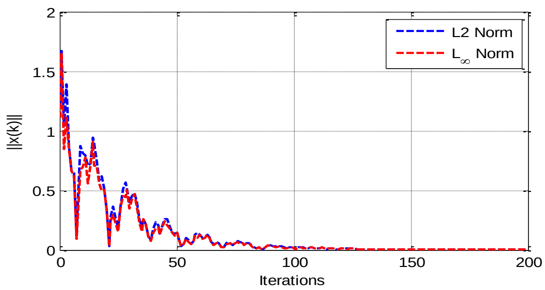

Fig.7. State norms under parametric uncertainty

Example 3. Let us examine the following uncertain system:

$$x_{(k+1)} = (S+\delta S(k))_i x_{(k)} + (S+\delta S(k))_{id} x_{(k-d)} \quad (54)$$

$$(x_{k+1})_{i=1} = \begin{bmatrix} 0.7 & 0 \\ 0.08 & 0.95 \end{bmatrix} x_k + \begin{bmatrix} 0.15 & 0 \\ -0.1 & -0.1 \end{bmatrix} x_{(k-26)}$$

$$(x_{k+1})_{i=2} = \begin{bmatrix} 0.7 & 0 \\ 0.08 & 0.9 \end{bmatrix} x_k + \begin{bmatrix} 0.14 & 0 \\ -0.1 & -0.1 \end{bmatrix} x_{(k-26)}$$

In whitch considering the uncertainty matrics (30)

$$Q_1 = \begin{bmatrix} 0.05 \\ 0 \end{bmatrix} . M_{a1} = [0.2 \quad 0.3] . M_{d1} = [-0.1 \quad -0.1]$$

$$Q_2 = \begin{bmatrix} 0.05 \\ -0.02 \end{bmatrix} . M_{a2} = [0 \quad -0.1] \; M_{d2} = [-0.3 \quad -0.2]$$

In [39]–[40], the maximum allowable delay d_2 ensuring the asymptotic stability of the system was reported for specific values of the minimum delay d_1. In order to provide a fair benchmark, these results are also referenced here for comparison. In our study, since the delay is assumed to be constant and precisely known, the corresponding values of d_2 from [39]–[40] are used only as reference points to highlight the improvement and reduced conservatism achieved by the proposed method. The initial conditions are given as:

$x(0) = [1 \quad 1]^T$

Table II. Maximum upper bound of d

	d_M
[39]	**16**
[40]	**22**
Theorem 4	**26**

Table II compares the maximum upper bound of the delay d obtained using Theorem 4 in this study. Accordingly, the results are presented as follows.

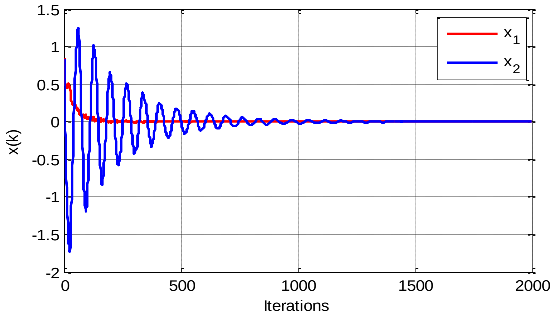

Fig.8. States of the system for for Example 3

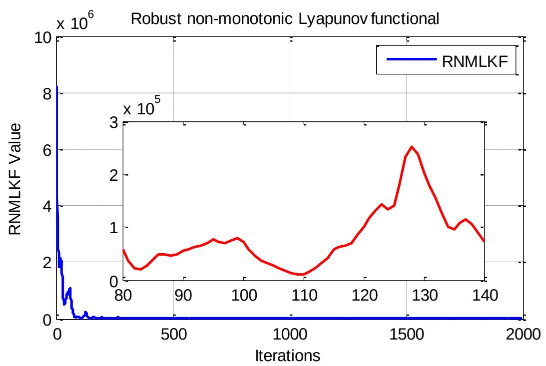

Fig. 9. Proposed RNMLKF for Example 3

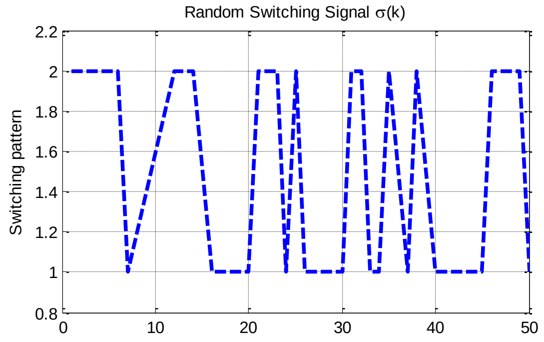

Fig.10. Switching pattern between subsystems

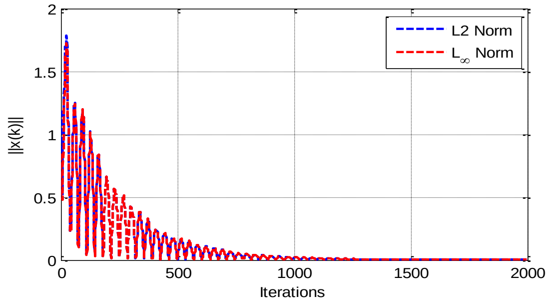

Fig.11. State norms under parametric uncertainty

5. Conclusion

In this paper, a novel non-monotonic Lyapunov–Krasovskii theorem is developed for the stability analysis of time-delay switching systems. Unlike classical approaches, this method permits the Lyapunov–Krasovskii function to increase temporarily at certain steps, provided that its overall trend remains decreasing. The concept of a "non-monotonic step" defines the maximum allowable number of such temporary increases; enlarging this step reduces conservatism but incurs higher computational cost. Numerical results demonstrate that integrating the non-monotonic structure with Abel's lemma substantially mitigates conservatism. Additionally, a robust non-monotonic stability theorem is derived for systems with uncertain delays, enlarging the feasible stability region relative to existing methods. The proposed approach ensures satisfactory stability performance, guaranteeing system convergence to equilibrium despite subsystem switching and the presence of unknown delays.

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