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A Comparative Study of Reordering Techniques for Consecutive Numbering in Power Grids

Mohammad Shahraeini[1,*]

[1] *Department of Electrical Engineering, Golestan University, Golestan , Iran*

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ABSTRACT

The structure and ordering of nodes in power grids significantly affect the performance of algorithms used for simulation, optimization, and visualization. While synthetic generation of power grids has gained momentum in both transmission and distribution domains, limited research has addressed how node indexing impacts matrix sparsity patterns and computational performance. This paper presents a comprehensive study of node reordering techniques aimed at achieving consecutive numbering in complex power networks. Four categories of methods are explored: heuristic approaches, optimization-based strategies, graph-theoretic algorithms, and artificial intelligence (AI)-driven models. We revisit existing heuristics and propose a novel metaheuristic optimization method that maximizes diagonal density across multiple bandwidth levels. For radial distribution grids, we develop a graph-based method using depth-first search (DFS) that aligns node numbering with physical structure. In addition, we evaluate the feasibility of using convolutional and feedforward neural networks to learn reordering patterns from data. Despite training on thousands of synthetic graphs, standard AI models fail to produce valid permutations, highlighting the need for permutation-aware architectures. Extensive experiments on synthetic transmission and distribution networks confirm that method selection should align with network topology. Optimization excels in meshed systems, while graph-based DFS is ideal for tree-like networks. A comparative analysis is provided to guide future applications and research in this space.

1. Introduction

Understanding the topological structure of power systems is fundamental to a wide range of applications, from network modeling and simulation to visualization and optimization. In recent years, the representation of power grids as complex graphs has become a widely accepted abstraction, enabling deeper insights into their structural properties and facilitating advanced computational methods. One specific area of interest is the ordering of nodes in power network graphs, which significantly impacts the interpretability and numerical performance of various algorithms, particularly those involving matrix-based formulations such as load flow, state estimation, and matrix factorization. The way nodes are indexed directly affects the sparsity pattern and diagonal compactness of matrices such as the adjacency matrix. As a result, effective node reordering strategies that align with the network's physical or logical structure can greatly enhance both analytical clarity and computational efficiency [1]-[5]. This work extends the methodology and findings originally introduced in [1], where early reordering techniques were proposed and evaluated primarily for transmission networks.

1.1. Complex Power Networks

Power systems are large-scale, interconnected infrastructures whose operation, planning, and optimization rely on both physical laws and topological

* Corresponding author
E-mail address: m.shahr@gu.ac.ir
https://orcid.org/0000-0003-2249-426X
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structure. In recent years, an increasing number of studies have adopted a graph-theoretic perspective to represent these systems as complex networks, enabling structural analysis and comparative studies across different regions and voltage levels [3]–[4], [14], [17].

In this representation, the network is modeled as an undirected graph $G=(V,E)$, where the set of vertices V corresponds to electrical buses, and the set of edges E denotes the physical interconnections, typically transmission lines or distribution feeders. Several studies have investigated the structural features of such graphs, including degree distributions, connectivity, and clustering coefficients, highlighting distinct characteristics compared to random or social networks [2]–[7].

The importance of these structural properties extends beyond pure topology. In particular, the way nodes are indexed directly affects the pattern of sparsity in associated matrices such as the adjacency matrix, incidence matrix, or Jacobian matrices used in power flow analysis. These patterns influence the performance of numerical algorithms such as LU factorization [8], preconditioning [9], and sparse matrix solvers [10]. Moreover, the interpretation of these matrices in visualization tools and data-driven models is often improved when node ordering reflects the network's physical layout or hierarchical structure [11].

While the structural complexity of transmission networks has been widely explored [3], [12]-[25], distribution networks (which are often radial and tree-like in nature) have distinct topological features that warrant specialized analysis and tailored reordering strategies [4]-[5], [26]-[30]. This duality in power system structure provides the foundation for the reordering approaches developed and compared in this study.

1.2. Research Gap and Research Motivation

In recent years, there has been growing interest in generating realistic power network topologies known as synthetic transmission grids (STGs) and synthetic distribution grids (SDGs). A number of studies (such as [3] for STGs and [4]–[5] for SDGs) have introduced scalable models for constructing synthetic power grids (SPGs) that replicate the statistical, structural, and electrical characteristics of real systems, enabling large-scale studies without access to utility-sensitive data.

While these developments have substantially advanced synthetic grid generation, much less attention has been devoted to the role of node ordering within these networks. The way buses are numbered directly affects the sparsity pattern of adjacency and incidence matrices, which in turn influences matrix-based computations, solver efficiency, storage formats, power-flow approximations, and even visualization methods such as Kirk or geographic layouts. Despite its practical relevance, systematic investigation of node reordering in power networks remains limited. Existing efforts (most notably [1]) focus primarily on heuristic and optimization-based reorderings, applied mainly to meshed transmission systems, and do not explore broader algorithmic alternatives or their behavior across different grid types.

Furthermore, transmission and distribution networks exhibit fundamentally different structural properties. Meshed transmission systems require ordering strategies that balance loop structures and connectivity, whereas radial distribution networks benefit from branch-aligned or depth-consistent numbering. Current literature does not provide a unified comparison of methods tailored to these distinct topologies, nor does it explore modern approaches such as graph-theoretic or AI-based reorderings.

This work addresses these gaps by systematically examining both existing reordering techniques (heuristic and optimization-based) and newly proposed methods (graph-based DFS reordering and learning-based models), evaluating their performance on STGs, SDGs, and real distribution feeders. The goal is to provide a comprehensive and topology-aware assessment of reordering strategies for power-system graphs.

1.3. Research Aims and Research Contribution

The overarching aim of this study is to systematically analyze, benchmark, and enhance node reordering strategies for power system graphs, with the goal of achieving numbering schemes that are both consecutive and structurally coherent. Improving node ordering has direct implications for matrix compactness, visualization clarity, and computational efficiency; factors that influence a wide range of power-system analysis and simulation tasks. To accomplish this aim, the paper makes the following key contributions:

1. A unified formulation and review of existing reordering methods for power network graphs, covering both meshed transmission systems and radial distribution feeders. The study consolidates heuristic [3] and optimization-based [1] approaches that have previously been explored in the literature and clarifies their strengths, assumptions, and limitations.
2. Introduction of new structure-aware reordering strategies, including a graph-theoretic DFS-based method specifically designed for radial distribution networks. This method leverages centrality-guided branch expansion and is shown (through both quantitative metrics and visual Kirk representations) to produce near-ideal consecutive numbering for synthetic radial grids.
3. Extension of performance evaluation metrics, including detailed diagonal density measures (D1–D5), execution-time analysis, and topology-sensitive comparisons across network sizes from 25 to 100 nodes. These metrics enable consistent assessment of existing and newly proposed approaches across STGs, SDGs, and real IEEE feeders.
4. Comprehensive benchmarking on both synthetic and real radial systems. While real IEEE feeders already possess structurally aligned numbering due to their historical development, the proposed DFS-based approach demonstrates substantial improvements on SDGs. The Kirk visualizations further illustrate these structural benefits.
5. Investigation of AI-based reordering models, including convolutional and fully connected neural architectures trained on large synthetic datasets. The study demonstrates the fundamental limitations of standard neural

networks in producing valid permutation outputs and identifies why such architectures struggle with discrete node-renumbering tasks. These findings motivate the need for more expressive approaches (such as GNNs) for future research in learning-based reordering.

Collectively, these contributions provide a systematic and topology-aware evaluation of node reordering strategies, integrating heuristic, optimization-based, graph-theoretic, and AI-driven perspectives. The resulting insights establish practical guidance for selecting appropriate methods across different classes of power system networks and applications.

Paper Structure

The remainder of this paper is organized as follows. Section 2 presents the foundational concepts and notations related to power network graphs, including their structural properties, representation methods, and diagonal density metrics. It also offers a comparative statistical analysis of both transmission and distribution test systems. Section 3 reviews heuristic- and optimization-based reordering strategies, including a rule-based approach previously proposed in [3] and an enhanced metaheuristic method designed to improve matrix compactness proposed in [1]. Section 4 investigates structure-aware and data-driven techniques, featuring a graph-based DFS algorithm for radial distribution grids and a critical evaluation of neural network models applied to the reordering problem. Section 5 synthesizes the results through a comparative analysis of all methods, assessing their effectiveness, input/output formats, and suitability for different grid types. Finally, Section 6 concludes the study by summarizing the main findings and outlining promising directions for future research.

2. Preliminaries

Power systems can be effectively modeled as graphs, enabling structural and algorithmic analysis using graph-theoretic tools. In this framework, electrical buses are represented as vertices and the lines connecting them as edges. This abstraction facilitates various applications, including reordering, clustering, and optimization. While many studies focus on high-voltage (HV) transmission networks, medium- and low-voltage (MV/LV) distribution systems can also be represented as graphs with distinct structural properties. In the following subsections, we review graph-based modeling characteristics of both transmission and distribution networks.

2.1. Power Graph Definition and Representation

In graph-based modeling of power systems, the network is represented as an undirected graph $G=(V,E)$, where V is the set of vertices representing electrical buses, and E is the set of edges corresponding to physical connections between buses. These edges typically represent lines or cables depending on the voltage level of the system. The graph can be described by its adjacency matrix $\mathbf{A}\in\{0,1\}_{n\times n}$; where $a_{ij}=1$ indicates the presence of an edge between nodes i and j, and $a_{ij}=0$ otherwise. The degree $\deg_i$ of a node i is defined as the number of edges incident to it:

$$\deg_i = \sum_{j=1}^{n} a_{ij} \tag{1}$$

2.1.1 Transmission Power Graph

In transmission-level modeling, the edges in graph G correspond to transmission lines, which connect high-voltage (HV) buses. These graphs are typically sparse and exhibit a meshed structure, meaning that there are multiple paths between some pairs of nodes. The average node degree in real-world transmission networks generally lies between 2.2 and 3.1, reflecting a relatively low connectivity per node, but sufficient redundancy for reliability [3]. Many widely used benchmark test systems (e.g. IEEE 14, 30, 57, 118, and 300 bus cases) are structured in this way [31].

2.1.2 Distribution Power Graph

Distribution systems, typically operating at medium-voltage (MV) and low-voltage (LV) levels, can also be modeled as graphs where vertices represent buses and edges represent distribution lines or cables. Unlike transmission networks, distribution grids often follow a tree-like (radial) structure, particularly in traditional designs where power flows unidirectionally from a central substation to downstream consumers. In such cases, the resulting graph is acyclic and connected (i.e., a tree) and therefore contains exactly $|E|=|V|-1$ edges [4]-[5].

Each vertex in a distribution graph is characterized by its load demand, power injection, or voltage magnitude, while each edge may be associated with electrical parameters such as resistance (used for cost analysis) or current capacity (used for operational constraints) [4]. The topological classification of distribution grids includes radial, parallel, ring, and interconnected types [5], with radial being the most widely adopted due to its simplicity and low installation cost.

2.2. Numbering of Power Graphs

In graph representations of power systems, node numbering refers to the indexing of vertices (buses) in the adjacency or incidence matrix. This numbering plays an important role in both the interpretation and the numerical treatment of graph-based algorithms. In standard test cases for transmission systems, such as the IEEE and MATPOWER benchmarks [31], nodes are often ordered in a manner that aligns with geographical layout, substation hierarchy, or operational zones. While this numbering is not explicitly optimized for computational structure, it frequently leads to partially consecutive patterns in the adjacency matrix due to the underlying grid topology [3]-[5].

Interestingly, distribution systems (particularly those modeled as radial networks) tend to exhibit even more consecutive numbering among their nodes. This behavior is a consequence of the tree structure common in distribution grids, where nodes are sequentially connected along branching feeders. As a result, when these networks are indexed in a depth-first or level-order traversal pattern, their adjacency and incidence matrices often display naturally ordered node sequences. This inherent order in distribution graphs facilitates diagonal compactness and simplifies structural analysis, making them especially well-suited for methods that exploit matrix sparsity or reordering schemes [1], [4]-[5].

2.3. q-steps off-diagonal density

The structural organization of a power graph's adjacency matrix can offer useful insights into how closely interconnected the nodes are, especially in relation to their numerical ordering. A key measure introduced in [1] to capture this characteristic is the "*q*-steps off-diagonal density", which quantifies the extent to which edges are concentrated around the main diagonal of the adjacency matrix.

Given a binary adjacency matrix $\mathbf{A}\in\{0,1\}_{n\times n}$; where each entry a_{ij} denotes the presence or absence of an edge between node i and node j, the q-steps off-diagonal density evaluates the number of nonzero entries (i.e., edges) that fall within a specified band of width q along the matrix's main diagonal. More precisely, the index pairs (i,j) satisfying $|i-j|\leq q$ form the off-diagonal band region. This density measure is formalized as:

$$D_q = \frac{1}{|E|} \sum_{\substack{1\leq i,\, j\leq n \\ |i-j|\leq q}} a_{ij} \tag{2}$$

where, $|E|$ is the total number of edges in the graph.

A higher value of D_q indicates that a greater proportion of edges are clustered near the main diagonal, reflecting a tighter and more localized connectivity structure in terms of node indexing. Conversely, a low D_q value suggests a more dispersed arrangement of edges throughout the matrix, potentially pointing to a less structured or more randomly ordered node sequence.

This metric is particularly useful in assessing the effectiveness of node reordering techniques, which aim to improve diagonal compactness; a desirable property in various graph algorithms and matrix computations.

2.4. Observations on Node Numbering in Benchmark Power Networks

To further understand how node numbering patterns manifest in benchmark power networks, we analyze several standard IEEE test cases by quantifying the off-diagonal density across multiple bands [31]. For each test system, the percentage of adjacency matrix entries falling within narrow bands along the main diagonal (denoted as D_q for q=1,2,3,4,5) is reported. These densities provide insight into how closely connected the nodes are in terms of their numeric ordering. A high D_1 or D_2 value suggests that the majority of connections exist between numerically adjacent nodes, reflecting an implicit ordering structure in the original data.

Table I. Off-diagonal density metrics for benchmark transmission networks.

Case Size	D_1	D_2	D_3	D_4	D_5	$D_{1\text{-}5}$
14	50	10	10	5	15	90
24	29.41	20.59	14.71	8.82	8.82	82.35
30	36.59	24.39	12.2	7.32	0	80.49
39	50	2.17	8.7	0	4.35	65.22
57	51.28	12.82	1.28	3.85	2.56	71.8
118	44.13	16.2	6.15	6.15	3.91	76.54
200	18.78	8.16	5.31	3.27	2.45	37.96
300	24.94	11.49	7.09	5.13	3.67	52.32
1354	0.06	0	0.23	0.18	0.29	0.76
1888	4.9	1.99	0.74	0.56	0.3	8.49
2383	9.04	4.37	1.66	1.35	1.11	17.53

Table I summarizes these metrics for a collection of widely used transmission test systems, indicating that many of them exhibit reasonably high diagonal density without any explicit reordering. This reveals that existing node indices, although not optimized, often follow partially structured patterns, especially in smaller or moderately meshed systems.

To complement this transmission-focused analysis, we now present results for several IEEE distribution network test cases form [31], shown in Table II. In addition to the off-diagonal density metrics D_1 through D_5 and $D_{1\text{-}5}$, we introduce a branching rate metric, denoted as BR, which quantifies the fraction of nodes in the graph with a degree of three or more [4]-[5]. This is formally defined as:

$$\text{BR} = \frac{\left|\left\{v_i \in V : \deg_i \geq 3\right\}\right|}{|V|} \times 100 \tag{3}$$

where, $\deg_i$ is the degree of node v_i. This measure reflects the extent of branching in the graph and is especially relevant for tree-like distribution networks.

Table II presents results for several standard IEEE radial distribution systems. Due to their tree structure and sequential feeder-based layout, these systems naturally exhibit very high D_1 densities (often above 85%), even without reordering. Notably, in cases like the IEEE 33-bus and 69-bus systems, nearly all edges fall within the first off-diagonal band. Additionally, the branching rates for these networks are relatively low, typically under 10% for most systems, which aligns with their radial architecture.

We summarize the following observations for distribution networks:

1) High diagonal density is common in radial systems, especially for small q, due to their inherent sequential and feeder-based node numbering;
2) Low branching rates reinforce this pattern, as fewer nodes deviate from the primary path or introduce additional connectivity complexity;
3) These structural tendencies distinguish them from meshed transmission systems and make them more amenable to traversal-based or heuristic reordering strategies.

Table II. Diagonal Density and Branching Rate for IEEE Distribution Test Cases

Case Size	BR	D_1	D_2	D_3	D_4	D_5	$D_{1\text{-}5}$
18	22.22	76.47	11.76	0	0	0	88.24
22	36.36	61.9	33.33	0	0	4.76	100
33	9.09	90.63	0	0	0	0	90.63

69	8.70	89.71	0	0	0	0	89.71
85	34.12	64.29	4.76	3.57	2.38	2.38	77.38
141	26.24	68.57	4.29	0.71	0	0.71	74.29

3. Review of Previous Methods for Consecutive Node Numbering in Power Graphs

This section reviews existing reordering strategies that have been used to improve the structural compactness of power network graphs. Two commonly applied approaches (heuristic [3] and metaheuristic optimization [1] methods) serve as the baseline techniques examined in this study. These methods operate primarily using local or partially informed structural cues and have shown effectiveness, particularly in transmission networks. The purpose of this section is to summarize these established methods so that they may serve as a point of comparison for the newly proposed techniques introduced in Sec. 4.

Algorithm 1. Heuristic node reordering based on shared edge incidence, adapted from [3].

Input: $\mathbf{A}$: adjacency matrix
Output: $\mathbf{A}^*$: renumbered adjacency matrix

```
procedure REORDER(A)
    Initialize: B* = [ 0 ]_{n×m} ; B = ADJ2INC(A)

    for j = 1 to M do
        for i = 1 to n do
            if b_ij = 1 then
                row ← B(i,:);  B(i,:) ← [0···0]
                for k = 1 to n do
                    if SUM(B*(k,:))= 0 then
                        B*(k,:) ← row
                        row ← [0···0]
                    end if
                end for
            end if
        end for
    end for
    A* ← ICN2ADJ(B*)
    return A*
end procedure
```

3.1. Heuristic-Based Reordering

Heuristic methods offer efficient, approximate solutions to computational problems by applying rule-based logic or local search strategies, without requiring exhaustive evaluation of all possibilities. In the context of node reordering for power network graphs, heuristics are particularly attractive due to their low computational cost and ability to reveal useful structure in large-scale systems [1], [3].

The heuristic reordering algorithm used in this study is originally proposed in [3] and is presented as Algorithm 1. Unlike most reordering schemes that operate directly on the adjacency matrix, this method is applied to the incidence matrix $\mathbf{B}\in\{0,1\}_{n\times m}$, where each column corresponds to an edge and each row to a node. In the binary incidence matrix, each column has exactly two entries with value 1, indicating the pair of nodes connected by the corresponding edge.

The algorithm begins by selecting a node with the minimum row sum in the incidence matrix, i.e., the node with the fewest edge connections; typically, a leaf or pendant node. This node is assigned the first number in the new ordering. Then, the algorithm iteratively selects the next node to be renumbered based on how many edges it shares with already-numbered nodes. For each unvisited node, a score is computed equal to the number of shared edges it has with the growing reordered set. Among the candidates with the highest score, the algorithm prefers the one with the lowest original degree, favoring nodes that extend the diagonal structure without introducing far-off entries.

This process continues until all nodes are renumbered. By working on the incidence matrix, the algorithm inherently takes into account how nodes are physically connected via edges, and builds a numbering sequence that keeps edge-related nodes close in index. As a result, when the reordered incidence or adjacency matrix is constructed, many of the nonzero entries are concentrated around the diagonal. This leads to a compact matrix structure that enhances the diagonal density measures discussed in Section 2.

Overall, this heuristic is effective for a wide range of power network topologies (particularly transmission networks) and its simplicity makes it a practical baseline for comparison against more complex or computationally intensive approaches.

3.2. Optimization-Based Techniques

In contrast to heuristic methods, optimization-based techniques aim to systematically search for a node permutation that maximizes a defined performance metric. These approaches often involve formal problem formulations (such as minimizing matrix bandwidth or maximizing diagonal density) and rely on either exact or approximate solvers to explore the solution space. In this subsection, we present two optimization strategies: the classical bandwidth minimization framework, and a proposed metaheuristic algorithm that directly targets diagonal compactness.

3.2.1 Bandwidth Minimization Problem

One of the most well-established optimization formulations for matrix reordering is the Bandwidth Minimization Problem (BMP), which seeks a permutation of the node indices that minimizes the spread of nonzero entries in the adjacency matrix away from the main diagonal. In the context of power system graphs, this corresponds to numbering nodes such that directly connected buses are assigned indices that are numerically close, leading to a compact matrix structure.

Formally, given an undirected graph $G=(V,E)$, the bandwidth of a vertex $v_i\in V$ is defined as the maximum absolute difference between the index of v_i and the indices of its neighbors:

$$b(v_i) = \max\left\{|i-j|, (v_i, v_j) \in E\right\} \tag{4}$$

where, (v_i,v_j) is the edge between node i and node j.

The graph bandwidth is then the maximum vertex bandwidth across all nodes:

$$B(G) = \max_{v_i\in V} b(v_i) \tag{5}$$

The objective of BMP is to find a node permutation that minimizes $B(G)$. This problem is known to be NP-complete, and thus heuristic and approximate methods (such as the Cuthill–McKee and Reverse Cuthill–McKee algorithms [32]) are commonly applied to obtain practical solutions in reasonable time.

Figure 1 illustrates the effect of applying BMP-based reordering on a sample graph. As shown, the original adjacency matrix has its nonzero entries scattered far from the diagonal, while the reordered version exhibits a tighter banded structure, improving compactness and diagonal density.

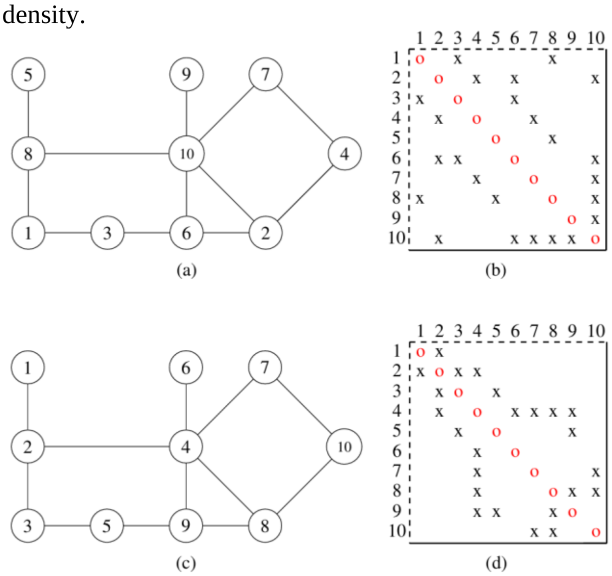


Fig. 1. The Effect of Bandwidth Minimization Problem on a Graph [1]. (a) Original Graph. (b) Adjacency Matrix of Original Graph - $B(G)$=8. (c) Reordered Graph. (d) Adjacency Matrix of Reordered Graph - $B(G)$=5.

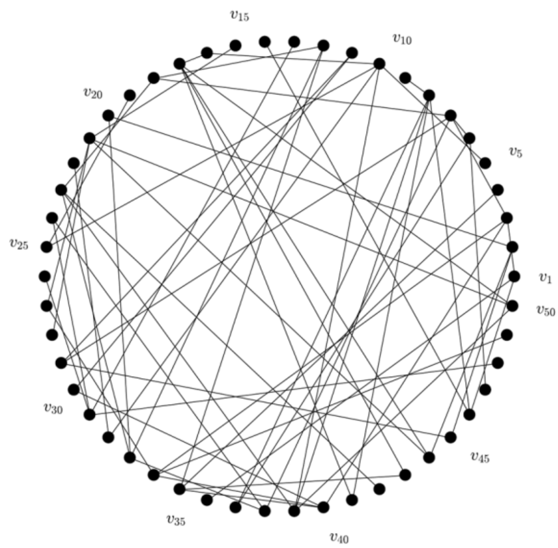
The diagonal density metrics introduced earlier in Section 2 offer a complementary perspective to the bandwidth concept. While $B(G)$ captures the worst-case spread of adjacency entries from the diagonal, the diagonal density D_q quantifies the concentration of edges within a band of width q. In this sense, both measures are aligned in assessing how closely connected nodes are indexed. A smaller bandwidth $B(G)$ typically correlates with higher values of D_q for small q, implying that most edges reside near the diagonal. However, unlike bandwidth, the density measure provides granular insight into how much of the adjacency structure is compact, rather than focusing solely on the furthest outliers. Therefore, D_q can be viewed as a more statistically descriptive counterpart to $B(G)$ in the analysis of reordering effectiveness.

Algorithm 2. Simple genetic algorithm

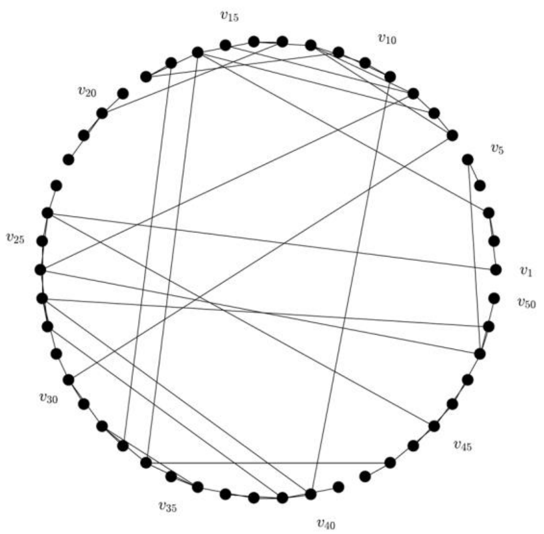
Initialize:
$t \leftarrow 0$
RANDOMIZE$(P(t=0))$
EVALUATE$(P(0))$ by Fitness Function
while $\neg$StopCondition **do**
    $P_p(t) \leftarrow$ SELECTION$(P(t))$
    $P_c(t) \leftarrow$ CROSSOVER$(P_p(t))$
    $P_m(t) \leftarrow$ MUTATE$(P_c(t))$
    EVALUATE$(P_m(t))$
    $P(t+1) \leftarrow$ TRUNCATE$(P_m(t), P(t))$
    $t \leftarrow t+1$
end while

3.2.2 Metaheuristic-Based Reordering via Density Maximization

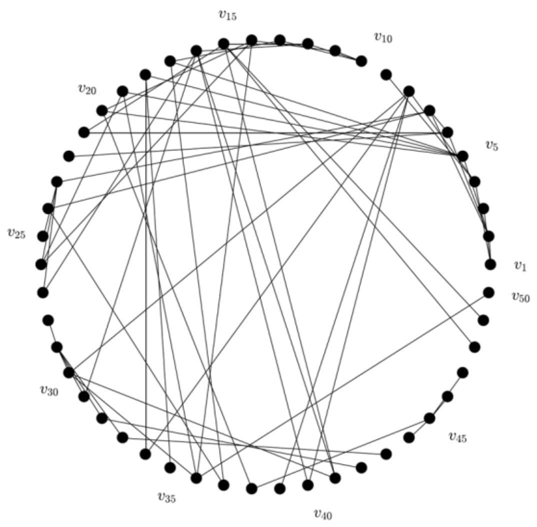
While the BMP focuses on the worst-case distance of connected nodes in the adjacency matrix, an alternative formulation is to directly maximize the overall diagonal density within a bounded band. This motivates an optimization approach aimed at finding a node permutation that increases the number of adjacency entries falling within a narrow region around the matrix diagonal.



(a)



(b)



(c)

Fig. 2. Kirk representations of 50-bus synthetic power grid before and after reordering ($D_{1\text{-}5}$=6.35). (a) Original grid. (b) Reordered using optimization method ($D_{1\text{-}5}$=26.67). (c) Reordered using heuristic method ($D_{1\text{-}5}$=28.57).

The core idea is to define a fitness function based on the cumulative density across multiple bands. Given the adjacency matrix $\mathbf{A}\in\{0,1\}_{n\times n}$, the optimization goal is to find a permutation vector $p\in\mathbb{N}^n$ such that the reordered matrix $\mathbf{A}(p,p)$ has the highest total diagonal density over a selected window. This can be interpreted as maximizing:

$$\sum_{q=1}^{Q} D_q(p) \qquad (6)$$

where, $D_q(p)$ is the percentage of adjacency entries within q-steps of the diagonal in the reordered matrix, and Q is a predefined band width.

To solve this problem, a metaheuristic search algorithm is proposed in [1], detailed in Algorithm 2. This method iteratively explores candidate permutations and evaluates their quality using the diagonal density fitness metric. In each iteration, a new candidate is generated by randomly swapping two elements of the current permutation. The new solution is accepted if it improves the fitness; otherwise, it may be retained based on a probability that allows limited uphill moves, helping the search escape local optima. This resembles a simulated annealing or hill-climbing framework.

Visual results for the reordering methods are shown in Figures 2a–2c, which illustrate the Kirk representations of the adjacency matrices. Figure 2a corresponds to the original matrix of a synthetic transmission network generated by method proposed in [3], where nonzero entries are widely scattered. Figure 2b shows the reordered matrix obtained using the optimization method described in Section 3.2, exhibiting a noticeably more compact diagonal structure. In contrast, Figure 2c presents the result of the heuristic approach proposed in Section 3.1, which also improves compactness but to a lesser degree than the optimization-based method.

Due to its flexible fitness formulation and stochastic nature, this approach is capable of discovering high-quality reordering schemes, especially in networks where heuristic methods may fail to capture deeper structural patterns.

4. Proposed Advanced Strategies for Node Reordering in Power Networks

High level description: This section introduces the new node-reordering methods developed in this study, motivated by the structural limitations identified in the existing heuristic and optimization-based approaches reviewed in Section 3. Two complementary directions are pursued. The first is a set of graph-theoretic strategies that explicitly exploit the inherent topology of power network graphs. In particular, we present a centrality-guided, depth-first search (DFS) renumbering method tailored for radial distribution grids, which is shown to produce highly consecutive ordering and substantial diagonal compactness, especially in synthetic SDGs (see Fig. 4). The second direction investigates learning-based approaches, where neural network models are trained to approximate bus reordering using adjacency-matrix representations. This section describes the design, training, and observed limitations of these AI-based models, providing insights into the challenges of learning permutation-structured outputs.

Algorithm 3. Tree Renumbering Based on Betweenness Centrality and Deepest Branch Traversal

```
Input: Adjacency matrix A of a tree with n nodes
Output: Renumbered adjacency matrix A_new

Compute betweenness centrality
  B = centrality(A, betweenness)
r ← arg max B            ▷ Find root with highest betweenness
Initialize:
  visited[1...n] ← false,
  newOrder[1...n] ← 0,     counter ← 1
Construct adjacency list L from A

function DFS(u)
    visited[u] ← true
    newOrder[u] ← counter, counter ← counter +1
    N ← unvisited neighbors of u in L
    for all v ∈ N do
        subtreeSize[v] ← COUNTSUBTREESIZE(v, u)
    end for
    Sort N in descending order of subtreeSize
    for all v ∈ N do
        if visited[v] = false then
            DFS(v)
        end if
    end for
end function

function COUNTSUBTREESIZE(v, p)
    s ← 1
    for all w ∈ L[v] do
        if w ≠ p then
            s ← s+ COUNTSUBTREESIZE(w, v)
        end if
    end for
    return s
end function
DFS(r)
Initialize A_new ← 0_{n×n}
for i = 1 to n do
    for j = i + 1 to n do
        if A[i, j] = 1 then
            u ← newOrder[i], v ← newOrder[j]
            A_new[u, v] ← 1, A_new[v, u] ← 1
        end if
    end for
end for
return A_new
```

4.1. Graph-Theoretic Approaches

As mentioned in [1], while the optimization-based reordering strategies discussed in Section 3.2 have demonstrated promising results on meshed transmission networks, they tend to be less effective for radial distribution networks. This limitation stems from the inherently sparse and tree-like topology of distribution systems, where the majority of nodes are organized along sequential feeders with limited branching. In such cases, the metaheuristic search often converges to shallow local optima or yields minimal improvement in diagonal compactness. To address this, we turn to graph-theoretic techniques that explicitly exploit the tree structure of radial networks. By applying traversal algorithms such as depth-first search (DFS), it becomes possible to construct a naturally consecutive node ordering that preserves feeder hierarchy and leads to significantly higher diagonal density, even without iterative optimization.

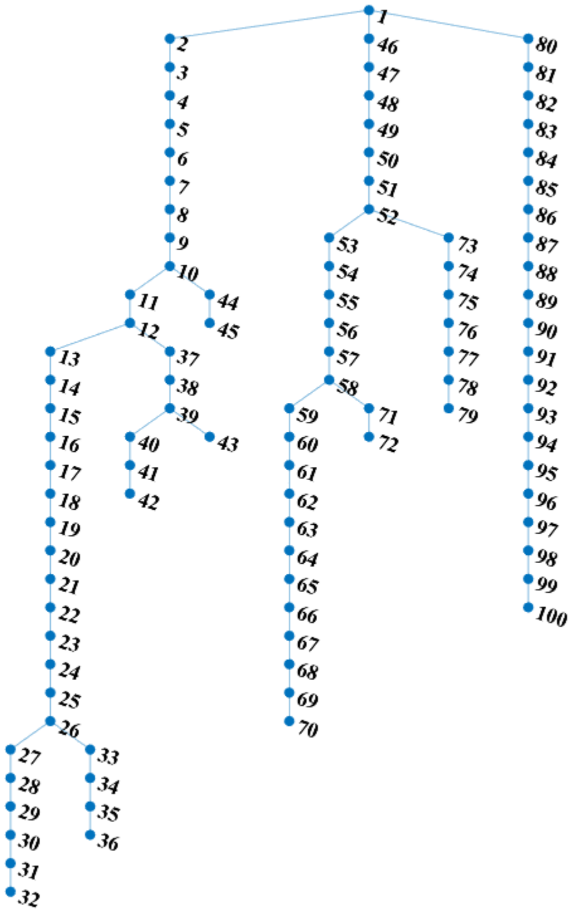


Fig 3. Reordered tree network using the graph-based DFS method, showing natural consecutive node numbering from center to leaves.

To directly leverage the tree topology found in radial distribution networks, we propose a graph-theoretic renumbering algorithm that builds on depth-first traversal principles. This method is particularly tailored for hierarchical structures, where consecutive labeling from central to peripheral nodes enhances diagonal compactness with minimal computational overhead.

The proposed algorithm, detailed in Algorithm 3, begins by computing the betweenness centrality of all nodes to identify the most topologically central node. This node is selected as the root of the traversal and is assigned the first index in the new numbering. The graph is then converted to an adjacency list representation to facilitate efficient recursive operations.

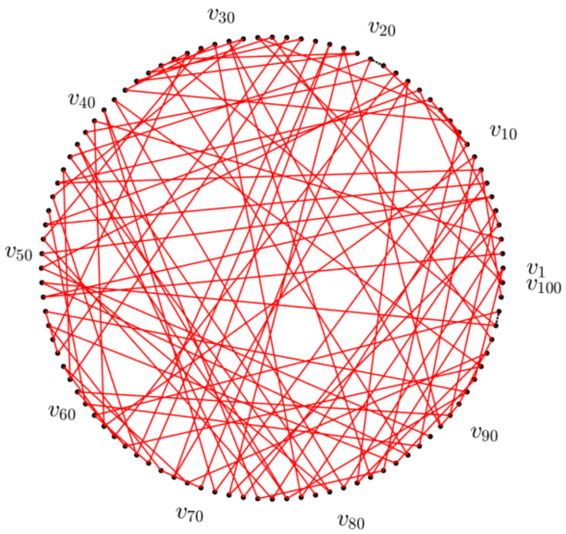
A global counter tracks the new labels assigned to each node. Using a DFS strategy, the algorithm traverses from the root, visiting unvisited neighbors in a prioritized order. To guide this traversal, the algorithm calculates the subtree size of each neighbor using a recursive function, and sorts them in descending order. This ensures that deeper branches (those contributing more nodes) are explored earlier, resulting in a label sequence that naturally progresses along the main paths of the tree.

Once all nodes have been visited and renumbered, the original adjacency matrix is reconstructed based on the new node indices. The resulting matrix exhibits a highly compact diagonal structure, where node numbers increase gradually from the center to the leaf nodes, faithfully reflecting the network's topological depth. This approach is deterministic, lightweight, and structure-aware, making it particularly well-suited for large-scale distribution grids where optimization-based methods are often inefficient or ineffective. As demonstrated in Section 5, the diagonal density achieved by this graph-theoretic method rivals or even surpasses that of more computationally intensive techniques when applied to radial networks.

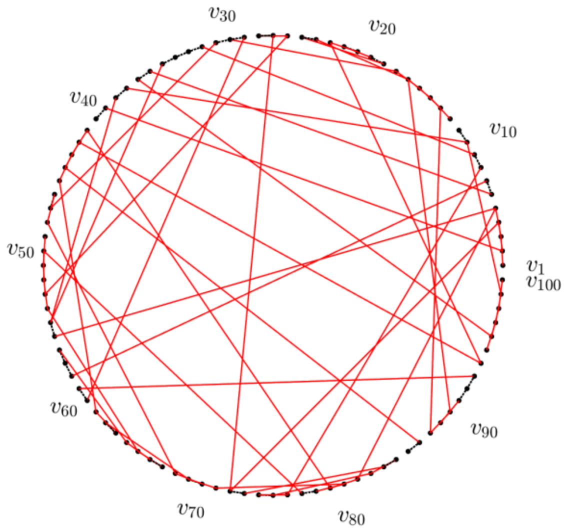
To evaluate the effectiveness of the proposed reordering techniques on radial topologies, we apply both the optimization-based method (Section 3.2) and the graph-theoretic DFS-based method (Section 4.1) to the tree-shaped network originally shown in Fig. 5 of [1]. The results of this comparison are summarized in Table III. The first row represents the diagonal density of the original numbering, which exhibits a very sparse structure, with only 11.11% of the adjacency entries located within five steps of the diagonal. After applying the optimization method, the density improves to 63.64%, primarily concentrated in the first two bands (see Fig. 5 of [1]). However, the DFS-based method achieves a significantly higher diagonal density of 92.93%, with over 91% of the entries located immediately adjacent to the diagonal. This dramatic improvement is also visually evident in Fig. 3, which shows the reordered version of the tree graph using the graph-based approach. These results confirm that structure-aware methods are far more effective for tree topologies than general-purpose optimization, both in terms of density and interpretability.

Table IV. Summary Comparison of Reordering Methods

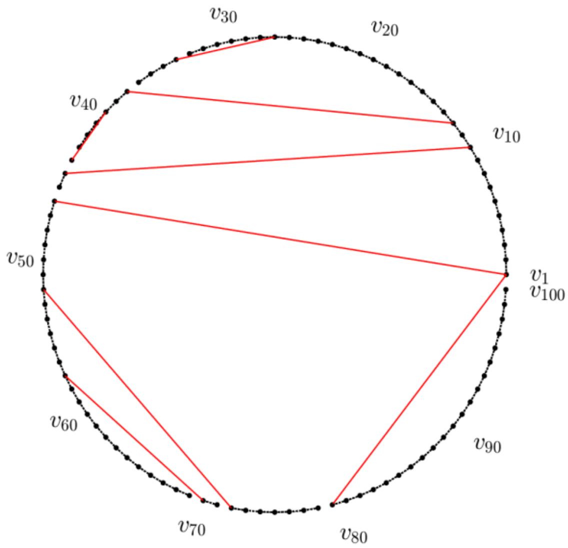
Method	Category	Input Type	Output Type	Accuracy (D_1–D_5)	Complexity	Suitable For
Heuristic (Sec. 3.1)	Rule-based	Incidence Matrix	Permutation	Medium	Low	Transmission (Fast)
Optimization (Sec. 3.2)	Metaheuristic	Adjacency Matrix	Permutation	High	Moderate-High	Transmission (Accurate)
Bandwidth Minimization (Sec. 3.2.1)	Optimization	Adjacency Matrix	Permutation	Moderate	Moderate	Transmission
Proposed Graph-based DFS (Sec. 4.1)	Topology-Aware	Adjacency List	Permutation	Very High	Very Low	Distribution (Tree)
Proposed CNN (Sec. 4.2.1)	Data-driven	Incidence Matrix	Matrix / Vector	Low	High	Not effective
Proposed Feedforward NN (Sec. 4.2.2)	Data-driven	Upper Adjacency	Permutation	Low	High	Not effective



(a)



(b)



(c)

Fig 4. Kirk representation of three different version of generated SDG shown in Fig. 3; (a): Initial version (b): Reordered by optimization, and (c): Proposed graph-based DFS method.

To further demonstrate the effectiveness of the proposed reordering techniques on radial topologies, The Kirk representations of the initial numbering, the optimization result, and the DFS-based reordering (shown in Fig. 3) are provided in Fig. 4. For enhanced visual clarity, black dashed chords indicate connections between consecutively numbered buses, whereas solid red chords represent non-consecutive connections. In the initial version (Fig. 4a), numerous red chords appear near the center of the plot, illustrating substantial deviation from a sequential branch expansion. The optimization-based approach (Fig. 4b) reduces the number of these red nonlocal connections; however, a noticeable gap still remains between its output and the structured numbering typically observed in real radial distribution feeders. In contrast, the proposed DFS-based renumbering (Fig. 4c) produces a nearly ideal consecutive ordering, leaving only eight red chords across the entire grid. This outcome is consistent with the quantitative diagonal density results in Table III, where the DFS method achieves over 92% density in the first five diagonals; substantially outperforming both the initial and optimization-based numberings. These findings reinforce that structure-aware, graph-theoretic approaches are particularly effective for radial networks, aligning the numbering closely with the underlying feeder expansion.

Table III. Diagonal density for original SDG and its reordered versions: by optimization method (§3.2), and proposed method (§4.1); shown in Figs 4-a to 4-c.

Method	D1	D2	D3	D4	D5	D1-5
Original	2.02	2.02	3.03	1.01	3.03	11.11
Optimization	30.3	33.33	0	0	0	63.64
Graph-based	91.92	0	0	1.01	0	92.93

In addition to the synthetic transmission networks analyzed previously, the proposed DFS-based renumbering method was also evaluated on several widely used IEEE radial distribution test feeders reported in Table III. These benchmark systems are typically constructed from real-world distribution feeders, where bus numbering follows the physical expansion of the network over time. As a result, their original numbering is already highly aligned with the underlying tree structure, leaving limited room for further improvement through algorithmic reordering. Accordingly, only marginal enhancements in the D1–5 diagonal compactness were observed; for instance, an increase from 77.3810% to 78.5714% for the IEEE 85-bus feeder and from 74.2857% to 76.4286% for the IEEE 141-bus feeder. In contrast, the benefits of the proposed DFS method become significantly more pronounced in synthetic or newly generated radial grids, where numbering is not historically inherited. As demonstrated in Fig. 4, the method achieves substantial improvements in diagonal concentration for such networks, confirming its suitability for synthetic grid generation, planning studies, and model standardization applications.

4.2. *AI-Driven Methods*

Recent advances in machine learning (particularly deep neural networks) have shown potential in learning complex graph patterns and transformations. Motivated by these developments, we investigated whether neural networks could learn to automatically reorder nodes in radial distribution networks to achieve a compact adjacency matrix structure. The goal was to train models that, given a representation of the original graph, could predict a new node order consistent with the high-diagonal-density structure produced by the graph-theoretic method described in Section 4.1.

To evaluate this possibility, we constructed a dataset of 12,000 synthetic distribution grids, each with 20 nodes, generated using the method proposed in [4]. These graphs were then renumbered using the DFS-based reordering technique from Section 4.1 to obtain the ground-truth “consecutively numbered” version. The data was split into 10,000 training and 2,000 validation samples. We tested both convolutional neural networks (CNNs) and fully connected feedforward networks, using various input and output encodings. Despite extensive experimentation, we found that both approaches consistently failed to produce correct reordering outputs.

4.2.1 *Convolutional Neural Network Approach*

In the CNN-based setup, we represented each input graph using its incidence matrix $\mathbf{B}\in\{0,1\}_{n\times(n-1)}$, which serves as a binary image capturing edge connectivity. Since renumbering the graph alters the row order of the incidence matrix, our initial idea was to train the CNN to map the original **B** to the reordered incidence matrix **B***. This was formulated as a supervised learning task with the CNN output being a binary image of the same dimensions $n\times(n-1)$.

However, this formulation failed to converge meaningfully. Predicted outputs were not valid incidence matrices: many columns had only one or more than two ones, violating the basic definition of incidence in trees. To address this, we reformulated the problem by keeping the input as the incidence matrix **B**, but changing the output to the permutation vector of size n, representing the target node ordering (i.e., a permutation of integers from 1 to 20). Despite trying various CNN architectures and training options, the model did not generalize well. The predicted vectors frequently contained repeated or missing values, and even when rounding was applied post-prediction, they failed to form valid permutations.

4.2.2 *Feedforward Neural Network Approach*

In an alternative configuration, we explored fully connected feedforward neural networks, aiming to reduce input dimensionality and focus on sequential patterns. Each input graph was represented by the upper triangular part of its adjacency matrix, flattened into a vector of size $\frac{n(n-1)}{2}=190$ for n=20. The corresponding output was again the permutation vector indicating the target node order.

We tested a variety of feedforward architectures with different numbers of hidden layers and neuron sizes. The network was trained in a regression setting, using the permutation vector as a continuous target and applying RMSE loss. However, this approach also failed to learn valid permutations. The predicted outputs often included duplicate or negative values, and rounding post-training did not correct these errors. Furthermore, even when the training RMSE dropped to low values, the validation error remained high and the final predictions were not valid reordering vectors.

4.2.3 *Limitations and Future Directions of AI Methods*

Although neural networks have shown strong performance in many power system applications, conventional architectures such as convolutional networks and fully connected models are fundamentally misaligned with the structure of the node-reordering problem. The task requires producing a valid permutation—a discrete, one-to-one mapping—yet standard neural models operate in continuous spaces, lack permutation equivariance, and do not naturally enforce bijectivity. As a result, they struggle with constraints such as uniqueness, ordering consistency, and combinatorial feasibility, regardless of training data volume or model size.

These limitations point toward more expressive families of models. In particular, graph neural networks (GNNs), message-passing architectures, and permutation-equivariant frameworks (e.g., DeepSets, graph transformers, or pointer-network-style decoders) are structurally compatible with graph reindexing tasks. Such models inherently respect graph symmetries, can learn neighborhood-aware transformations, and can generate discrete outputs through attention-based selection or learned ranking mechanisms. While the exploration of these architectures is beyond the scope of the present work, they represent a promising and mathematically suitable direction for future research.

Despite testing a wide range of conventional architectures, both CNN- and feedforward-based models consistently failed to generate valid node reorderings. CNNs struggled to predict a correct incidence matrix or permutation vector, even when thousands of labeled samples were used for training. Likewise, feedforward networks (even with well-tuned regression settings) produced outputs with duplicated indices, missing nodes, or non-permuted sequences, and displayed clear overfitting between training and validation.

These findings collectively indicate that standard neural network structures are not well suited for permutation-generating graph problems when framed as raw regression or image-to-vector mapping. The strict structural constraints and combinatorial nature of the task require architectures that inherently model graph symmetries and discrete mappings—reinforcing GNN-based and permutation-equivariant designs as the most appropriate future direction.

5. Comparative Analysis of Reordering Methods

To consolidate insights from the previous sections, this part of the study presents a side-by-side evaluation of all reordering techniques explored in this work. Table IV summarizes the characteristics and relative strengths of six methods, including: spanning heuristic logic, optimization algorithms, structural graph-based techniques, and machine learning approaches. The comparison criteria include input/output types, expected accuracy based on diagonal density (D1–D5), computational suitability, and relative speed and generalization capacity across different network topologies.

A related observation arises from the evaluation of the AI-based approaches discussed in Section 4.2. While neural methods were initially considered as potential data-driven alternatives to conventional reordering strategies, their empirical limitations prevent them from contributing directly to the comparative performance results summarized in this section. Standard CNN and feedforward architectures were unable to generate valid permutation outputs, and therefore they do not appear in the quantitative runtime or diagonal-density comparisons. Their inclusion in this study serves a different purpose: to clarify why naïve neural formulations fail and to motivate the need for more structurally aligned models such as graph neural networks or permutation-equivariant architectures. As such, the comparisons presented in this section focus exclusively on the methods that reliably produce valid reorderings; namely the heuristic, optimization-based, and graph-theoretic DFS approaches.

Heuristic and optimization-based approaches are more suitable for meshed transmission systems due to their ability to exploit partial structure without full topology awareness. In particular, optimization techniques outperform heuristics in terms of accuracy but at the cost of higher runtime complexity. On the other hand, the graph-based DFS method proposed in Section 4.1 demonstrates excellent performance on radial distribution networks, achieving superior diagonal compactness with minimal computation. In contrast, AI-based models such as CNNs and feedforward neural networks failed to produce valid reorderings, despite extensive training on synthetic data, indicating their current limitations for such discrete permutation tasks.

6. Conclusion

This paper presented a comprehensive study of node reordering techniques for enhancing the structural compactness of power network graphs through consecutive numbering. By targeting the improvement of diagonal density in adjacency representations, we explored and evaluated a diverse range of approaches, spanning heuristic methods, optimization frameworks, graph-theoretic algorithms, and artificial intelligence.

Our findings reveal that reordering strategies must be selected in accordance with the underlying network topology. For transmission networks, which typically exhibit meshed structures, optimization-based techniques; particularly those maximizing diagonal density, achieve the highest performance in terms of matrix compactness. Heuristic methods provide a faster alternative with acceptable accuracy. For radial distribution networks, however, these methods are significantly less effective. In such cases, a structure-aware graph traversal algorithm based on depth-first search demonstrates remarkable performance, achieving over 90% diagonal density while preserving the natural flow of node indices in tree structures.

Additionally, we conducted extensive experiments using convolutional and feedforward neural networks trained on thousands of synthetic distribution grids. Despite varying input representations and output formats, none of the tested models were able to generalize the reordering process or produce valid permutation vectors, highlighting the challenges of applying standard AI architectures to discrete graph permutation tasks.

The findings of this work offer clear guidance for selecting appropriate reordering strategies across transmission and distribution networks. They also point toward several promising avenues for future research, particularly in advancing learning-based approaches. More expressive architectures (such as graph neural networks, permutation-equivariant models, or hybrid frameworks that integrate structural heuristics with machine learning) may overcome the inherent limitations observed in standard neural networks. Exploring these directions could enable AI-driven methods to play a more effective role in node renumbering and related graph-structured tasks in power system analysis.

7. References

[1] M. Shahraeini, "Diagonal Density: Optimization Technique for Consecutive Numbering in Synthetic Power Grids," *2024 14th Smart Grid Conference (SGC)*, Tehran, Iran, Islamic Republic of, 2024, pp. 1-5, doi: 10.1109/SGC64640.2024.10982830.

[2] G. A. Pagani and M. Aiello, "The power grid as a complex network: a survey," *Physica A: Statistical Mechanics and its Applications*, vol. 392, no. 11, pp. 2688–2700, 2013.
[3] M. Shahraeini, "Modified Erdős–Rényi Random Graph Model for Generating Synthetic Power Grids," in *IEEE Systems Journal*, vol. 18, no. 1, pp. 96-107, March 2024, doi: 10.1109/JSYST.2023.3339664.
[4] M. Shahraeini, "A Comprehensive Approach to Synthetic Distribution Grid Generation: Erdős–Rényi to Barabási-Albert," *AUT Journal of Electrical Engineering*, vol. 57 no. 1, July 2024, doi: 10.22060/eej.2024.23143.5591.
[5] M. Shahraeini, "An algorithm for generating synthetic distribution grids based on Erdős–Rényi random graph model," in *2023 13th Smart Grid Conference (SGC)*. IEEE, 2023, pp. 1–8.
[6] F. Giacomarra, G. Bet, and A. Zocca, "Generating synthetic power grids using exponential random graph models," *PRX Energy*, vol. 3, no. 2, p. 023005, 2024.
[7] Y. Liu, A. Ştefanov and P. Palensky, "Generating Large-Scale Synthetic Communication Topologies for Cyber–Physical Power Systems," in *IEEE Transactions on Industrial Informatics*, vol. 20, no. 11, pp. 13463-13472, Nov. 2024.
[8] C. Hildebrandt, B. C. Karatas, J. G. Møller, and H. Jóhannsson, "Evaluation of Factorization Methods for Thévenin Equivalent Computations in Real-Time Stability Assessment," in *Proc. 2018 Power Systems Computation Conference (PSCC)*, pp. 1-7, Jun. 2018, doi: 10.23919/PSCC.2018.8442893.
[9] Y. Liu, A. K. Singh, J. Zhao, A. P. S. Meliopoulos, B. Pal, M. A. bin M. Ariff, T. Van Cutsem, M. Glavic, Z. Huang, I. Kamwa, L. Mili, A. S. Mir, A. Taha, V. Terzija, and S. Yu, "Dynamic State Estimation for Power System Control and Protection," *IEEE Transactions on Power Systems*, vol. 36, no. 6, pp. 5909-5923, Nov. 2021, doi: 10.1109/TPWRS.2021.3086202.
[10] X. Chen, "Numerically-Stable and Highly-Scalable Parallel LU Factorization for Circuit Simulation," *ACM Transactions on Design Automation of Electronic Systems*, vol. 27, no. 5, pp. 1-25, Oct. 2022. [Online]. Available: https://dl.acm.org/doi/10.1145/3508352.3549337.
[11] Y. Liu et al., "Dynamic State Estimation for Power System Control and Protection," *IEEE Transactions on Power Systems*, vol. 36, no. 6, pp. 5909-5923, Nov. 2021. (Includes topology-aware modeling aspects in state estimation for power grids.) doi: 10.1109/TPWRS.2021.3086202.
[12] B. A. Carreras, V. E. Lynch, I. Dobson, and D. E. Newman, "Critical points and transitions in an electric power transmission model for cascading failure blackouts," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 12, no. 4, pp. 985–994, 2002.
[13] M. Parashar, J. S. Thorp, and C. E. Seyler, "Continuum modeling of electromechanical dynamics in large-scale power systems," *IEEE Trans. Circuits Syst. I*, vol. 51, no. 9, pp. 1848–1858, 2004.
[14] Z. Wang, A. Scaglione, and R. J. Thomas, "Generating statistically correct random topologies for testing smart grid communication and control networks," *IEEE Trans. Smart Grid*, vol. 1, no. 1, pp. 28–39, 2010.
[15] S. Pahwa, C. Scoglio, and A. Scala, "Abruptness of cascade failures in power grids," *Sci. Rep.*, vol. 4, no. 1, Jan. 2014.
[16] J. Hu, L. Sankar, and D. J. Mir, "Cluster-and-connect: An algorithmic approach to generating synthetic electric power network graphs," in *Proc. 53rd Annu. Allerton Conf. Communication, Control, and Computing (Allerton)*, IEEE, 2015, pp. 223–230.
[17] S. H. Elyas and Z. Wang, "Improved synthetic power grid modeling with correlated bus type assignments," *IEEE Trans. Power Syst.*, vol. 32, no. 5, pp. 3391–3402, 2016.
[18] K. M. Gegner, A. B. Birchfield, T. Xu, K. S. Shetye, and T. J. Overbye, "A methodology for the creation of geographically realistic synthetic power flow models," in *Proc. IEEE Power and Energy Conf. Illinois (PECI)*, 2016, pp. 1–6.
[19] S. Ma, Y. Yu, and L. Zhao, "Dual-stage constructed random graph algorithm to generate random graphs featuring the same topological characteristics with power grids," *J. Mod. Power Syst. Clean Energy*, vol. 5, no. 5, pp. 683–695, Sep. 2017.
[20] S. Soltan, A. Loh, and G. Zussman, "A learning-based method for generating synthetic power grids," *IEEE Syst. J.*, vol. 13, no. 1, pp. 625–634, 2018.
[21] T. Xu, A. B. Birchfield, and T. J. Overbye, "Modeling, tuning, and validating system dynamics in synthetic electric grids," *IEEE Trans. Power Syst.*, vol. 33, no. 6, pp. 6501–6509, 2018.
[22] E. Schweitzer and A. Scaglione, "A mathematical programming solution for automatic generation of synthetic power flow cases," *IEEE Trans. Power Syst.*, vol. 34, no. 1, pp. 729–741, 2018.
[23] A. B. Birchfield, T. Xu, and T. J. Overbye, "Power flow convergence and reactive power planning in the creation of large synthetic grids," *IEEE Trans. Power Syst.*, vol. 33, no. 6, pp. 6667–6674, 2018.
[24] S. G. Aksoy, E. Purvine, E. Cotilla-Sanchez, and M. Halappanavar, "A generative graph model for electrical infrastructure networks," *J. Complex Netw.*, vol. 7, no. 1, pp. 128–162, 2019.
[25] M. Shahraeini, A. Alvandi, and S. Khormali, "Behavior analysis of random power graphs for optimal PMU placement in smart grids," in *Proc. 10th Int. Conf. Computer and Knowledge Engineering (ICCKE)*, IEEE, Oct. 2020.
[26] V. K. Krishnan, B. S. Palmintier, B. S. Hodge, E. T. Hale, T. Elgindy, B. Bugbee, M. N. Rossol, A. J. Lopez, D. Krishnamurthy, C. Vergara et al., "Smart-DS: Synthetic models for advanced, realistic testing: Distribution systems and scenarios," Nat. Renew. Energy Lab. (NREL), Golden, CO, Tech. Rep., 2017.
[27] E. Schweitzer, A. Scaglione, A. Monti, and G. A. Pagani, "Automated generation algorithm for synthetic medium voltage radial distribution systems," *IEEE J. Emerg. Sel. Topics Circuits Syst.*, vol. 7, no. 2, pp. 271–284, 2017.
[28] A. Trpovski, D. Recalde, and T. Hamacher, "Synthetic distribution grid generation using power system planning: Case study of Singapore," in *Proc. 53rd Int. Univ. Power Eng. Conf. (UPEC)*, IEEE, 2018, pp. 1–6.
[29] C. Mateo, F. Postigo, F. de Cuadra, T. G. San Roman, T. Elgindy, P. Dueñas, B.-M. Hodge, V. Krishnan, and B. Palmintier, "Building large-scale US synthetic electric distribution system models," *IEEE Trans. Smart Grid*, vol. 11, no. 6, pp. 5301–5313, 2020.
[30] R. Yan, Y. Yuan, Z. Wang, G. Geng, and Q. Jiang, "Active distribution system synthesis via unbalanced graph generative adversarial network," *IEEE Trans. Power Syst.*, 2022.
[31] R. D. Zimmerman, C. E. Murillo-Sánchez, and R. J. Thomas, "MATPOWER: Steady-state operations, planning, and analysis tools for power systems research and education," *IEEE Transaction on Power System*, vol. 26, no. 1, pp. 12-19, Feb. 2011.
[32] S. Stan and P. Bientinesi, "RCM++: Reverse Cuthill–McKee Ordering with Bi-Criteria Node Finder," *SIAM Journal on Scientific Computing*, vol. 46, no. 2, pp. A1234–A1255, 2024, doi: 10.1137/23M1556789.